OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS STANDARD LEVEL

COURSE COMPANION

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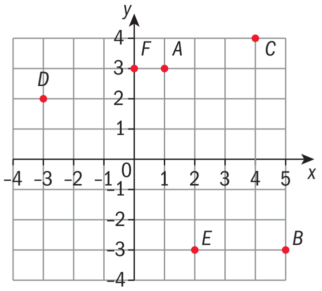
OXFORD

1 Functions

Answers

Skills check

1 a



b $A(0, 2)$, $B(1, 0)$, $C(-1, 0)$, $D(0, 0)$, $E(2, 1)$, $F(-2, -2)$, $G(3, -1)$, $H(-1, 1)$

2 a $4x + 3y = 4(4) + 3(6)$
$= 16 + 18$
$= 34$

b $z^2 - 3y = (-10)^2 - 3(6)$
$= 100 - 18$
$= 82$

c $y - z = 6 - (-10)$
$= 6 + 10$
$= 16$

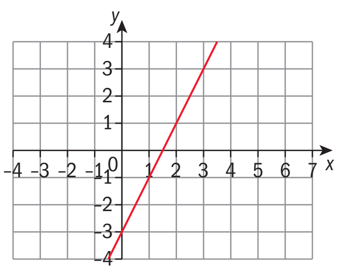
d $\frac{2x+5}{yz} = \frac{2(4)+5}{6\times(-10)}$
$= -\frac{8+5}{60}$
$= -\frac{13}{60}$

3 a $3x - 6 = 6$
$3x = 12$
$x = 4$

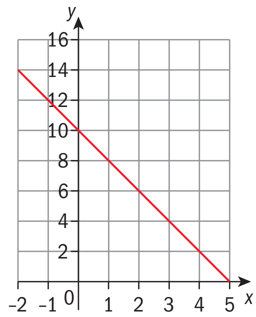
b $5x + 7 = -3$
$5x = -3 - 7 = -10$
$x = -2$

c $\frac{x}{2} + 6 = 11$
$x + 12 = 22$
$x = 10$

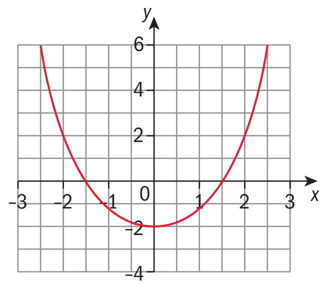
4 a



b



c



5 a $(x + 4)(x + 5) = x^2 + 5x + 4x + 20$
$= x^2 + 9x + 20$

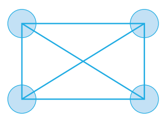
b $(x - 1)(x - 3) = x^2 - 3x - x + 3$
$= x^2 - 4x + 3$

c $(x + 5)(x - 4) = x^2 - 4x + 5x - 20$
$= x^2 + x - 20$

Investigation – handshakes

Represents one person. Represents one handshake

1 a

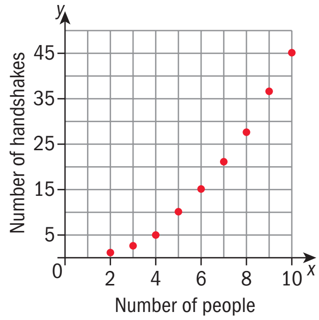


So 4 people require 6 handshakes.

b

Number of people	Number of handshakes
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45

c



d $H = \frac{1}{2}n(n-1)$

Exercise 1A

1 **a** Function. All x values are different

b Function. All x values are different

c Relation. The domain contains more than one 4

d Relation. The domain contains two ones

e Relation. The domain contains two -4s and two -3s

f Function. All x values are different.

2 **a** The domain is $\{0, 1, 2, 3, 4\}$

The range is $\{0, 1, 2\}$

It is a function because the domain has exactly one of each value.

b The domain is $\{-1, 0, 1, 2, 3\}$

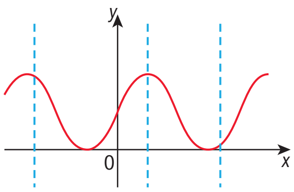
The range is $\{-1, 0, 1, 2\}$

Not a function as domain contains two -1s.

3 It is a function because the domain has exactly one of each value.

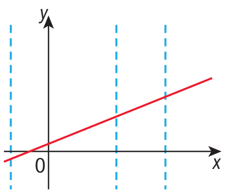
Exercise 1B

1 **a**



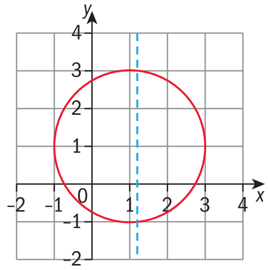
Function. Crosses only once

b



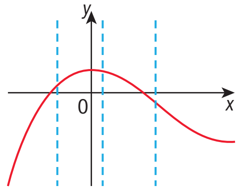
Function. Crosses only once

c



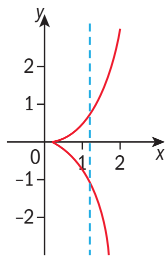
Relation. Crosses twice

d



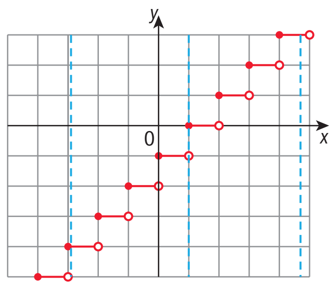
Function. Crosses only once

e



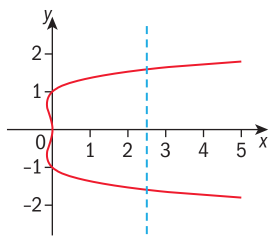
Relation. Crosses twice

f



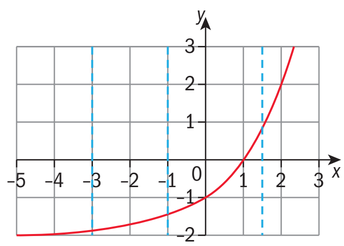
Function. Crosses only once

g



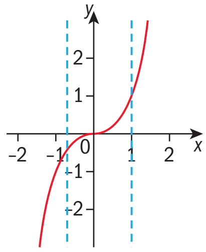
Relation. Crosses twice

h



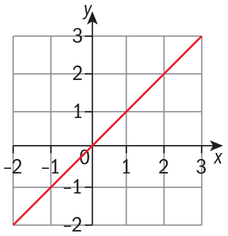
Function. Crosses only once

i

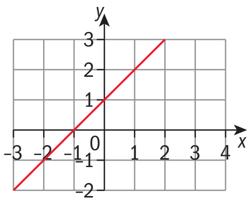


Function. Crosses only once

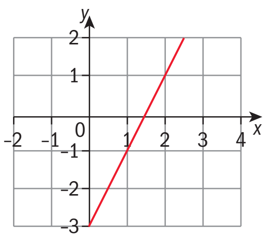
2 **a** $y = x$



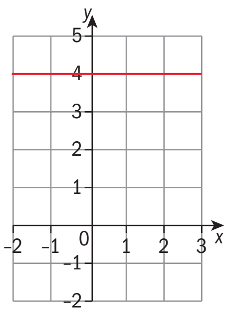
b $y = x + 2$



c $y = 2x - 3$



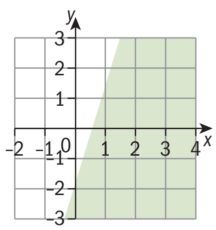
d $y = 4$



e Yes. A vertical line will only cross them once.

f No, vertical lines such as $x = 3$ are not functions.

3



Not a function as a vertical line crosses the region in many places

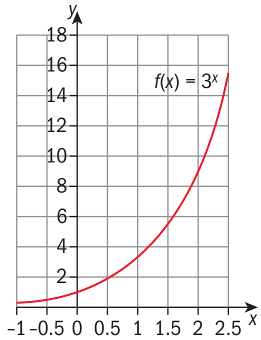
4 Manipulate the equation to make y the subject:

$y^2 = 4 - x^2, y = \pm\sqrt{4 - x^2}$

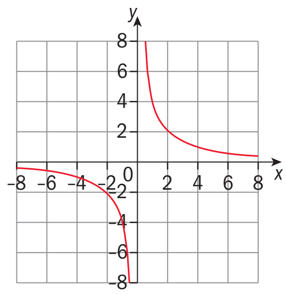
There are two possible values of y for any given x. For example, When $x = 1$, $y = \sqrt{3}$, $-\sqrt{3}$. The same value in the domain has two possible values in the range. Therefore $x^2 + y^2 = 4$ is not a function.

Exercise 1C

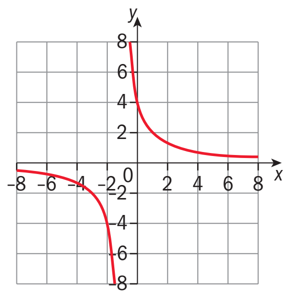
1 $y = 0$



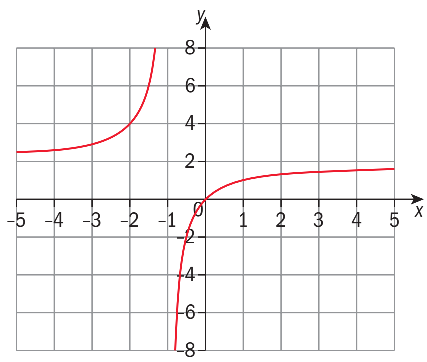
2 $y = 0$, $x = 0$.



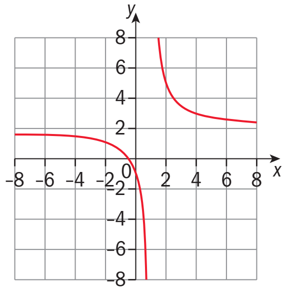
3 $y = 0$, $x = -1$.



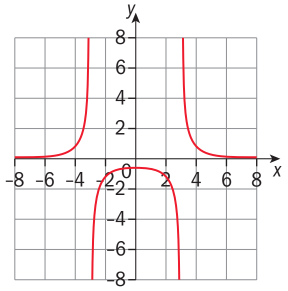
4 $y = 2$, $x = -2$.



5 $y = 2,\ x = 1.$



6 $y = 0,\ x = -3,\ x = 3.$



Exercise 1D

1 It is a function. Domain of $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ has no value repeated.

Range $\{1, 3, 6, 10, 15, 21, 28, 36, 45\}$.

2 Note that domain and range can be expressed in many ways.

a *Domain* $\{x : -4 < x \le 4\}$, Range $\{y : 0 \le y \le 4\}$

b *Domain* $\{x : -1 \le x \le 5\}$, Range $\{y : 0 \le y \le 4\}$

c *Domain* $\{x :\ -\infty < x < \infty\}$, Range $\{y :\ 0 \le y < \infty\}$

d *Domain*$\{x : -2 \ge x > 2\}$, Range $\{y : 3 \ge y \ge 4\}$

e *Domain*$\{x : -5 \le x \le 5\}$, Range $\{y : -3 \le y \le 4\}$

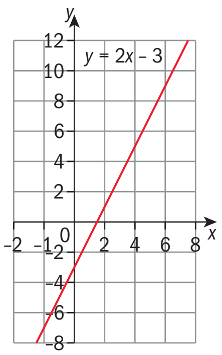
f *Domain* $\{x :\ -\infty < x < \infty\}$, Range $\{y :\ -1 \le y \le 1\}$

g *Domain*$\{x : -2 \le x \le 2\}$, Range $\{y : -2 \le y \le 2\}$

h *Domain*$\{x : -\infty < x < \infty\}$, Range $\{y : -\infty < y < \infty\}$

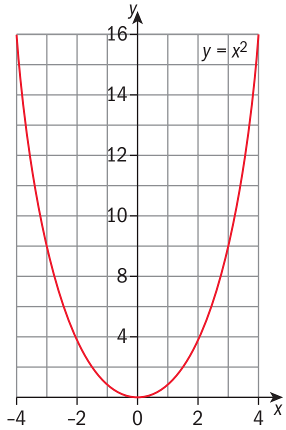
i *Domain* $x \in \mathbb{R},\ x \ne 1$, Range $y \in \mathbb{R},\ y \ne 0$.

3 **a**



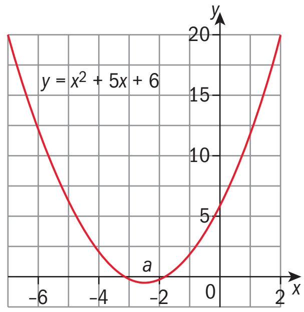
Domain $x \in \mathbb{R}$, Range $y \in \mathbb{R}$

b



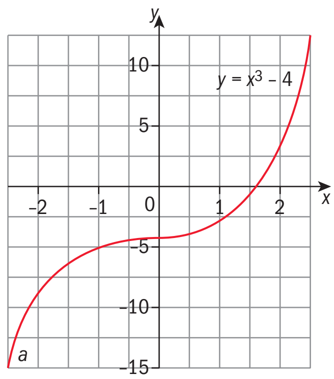
Domain $x \in \mathbb{R}$, Range $y \ge 0$

c



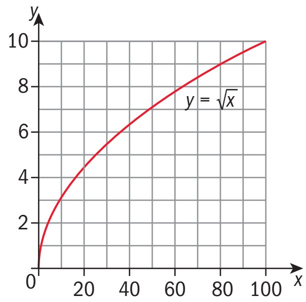
Domain $x \in \mathbb{R}$, Range $y \ge -0.25$

d



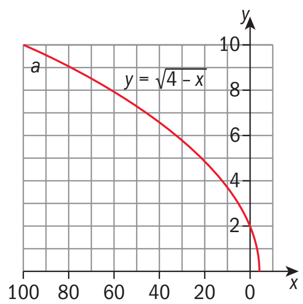
Domain $x \in \mathbb{R}$, Range $y \in \mathbb{R}$

e



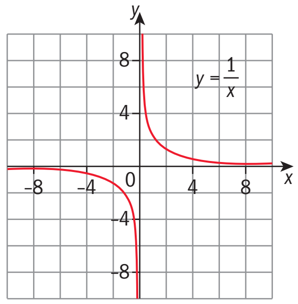
Domain $x \ge 0$, Range $y \ge 0$

f



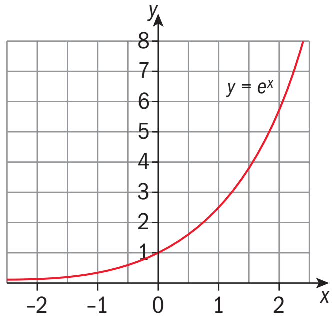
Domain $x \le 4$, Range $y \ge 0$

g



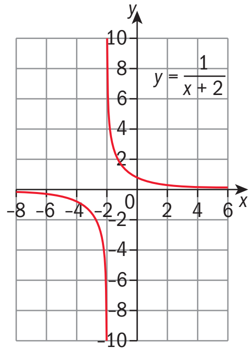
Domain $x \in \mathbb{R}$, $x \neq 0$, Range $y \in \mathbb{R}$, $y \neq 0$

h



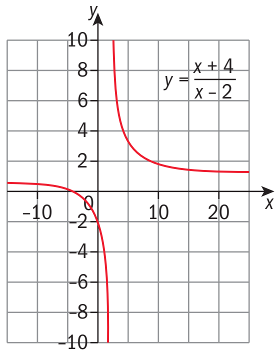
Domain $x \in \mathbb{R}$, Range $y > 0$

i



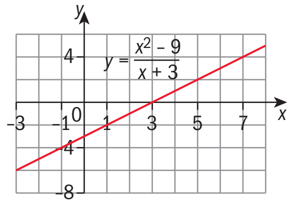
Domain $x \in \mathbb{R}$, $x \neq -2$, Range $y \in \mathbb{R}$, $y \neq 0$

j



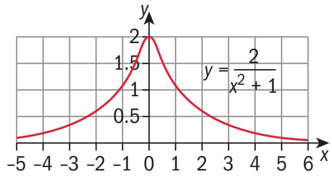
Domain $x \in \mathbb{R}$, $x \neq 2$, Range $y \in \mathbb{R}$, $y \neq 1$

k



Domain $x \in \mathbb{R}$, $x \neq -3$, Range $y \in \mathbb{R}$, $y \neq -6$

l



Domain $x \in \mathbb{R}$, Range $0 < y \leq 2$

Exercise 1E

1 a i $f(7) = 7 - 2 = 5$

ii $f(-3) = -3 - 2 = -5$

iii $f(\frac{1}{2}) = \frac{1}{2} - 2 = -1\frac{1}{2}$

iv $f(0) = 0 - 2 = -2$

v $f(a) = a - 2$

b i $f(3) = 3(7) = 21$

ii $f(-3) = 3(-3) = -9$

iii $f(\frac{1}{2}) = 3(\frac{1}{2}) = 1\frac{1}{2}$

iv $f(0) = 3(0) = 0$

v $f(a) = 3(a) = 3a$

c i $f(7) = \frac{1}{4} \times 7 = \frac{7}{4}$

ii $f(-3) = \frac{1}{4} \times -3 = -\frac{3}{4}$

iii $f(\frac{1}{2}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

iv $f(0) = \frac{1}{4} \times 0 = 0$

v $f(a) = \frac{1}{4} \times a = \frac{a}{4}$

d i $f(7) = 2(7) + 5 = 19$

ii $f(-3) = 2(-3) + 5 = -1$

iii $f(\frac{1}{2}) = 2(\frac{1}{2}) + 5 = 6$

iv $f(0) = 2(0) + 5 = 5$

v $f(a) = 2(a) + 5 = 2a + 5$

e i $f(7) = 7^2 + 2 = 51$

ii $f(-3) = (-3)^2 + 2 = 11$

iii $f(\frac{1}{2}) = (\frac{1}{2})^2 + 2 = 2\frac{1}{4}$

iv $f(0) = (0)^2 + 2 = 2$

v $f(a) = (a)^2 + 2 = a^2 + 2$

2 a $f(-a) = (-a)^2 - 4 = a^2 - 4$

b $f(a + 5) = (a + 5)^2 - 4 = a^2 + 10a + 25 - 4$
$= a^2 + 10a + 21$

c $f(a - 1) = (a - 1)^2 - 4 = a^2 - 2a + 1 - 4$
$= a^2 - 2a - 3$

d $f(a^2 - 2) = (a^2 - 2)^2 - 4 = a^4 - 4a^2 + 4 - 4$
$= a^4 - 4a^2$

e $f(5 - a) = (5 - a)^2 - 4 = 25 - 10a + a^2 - 4$
$= a^2 - 10a + 21$

3 a $g(x) = 3$, so $4x - 5 = 3$
$4x = 8$
$x = 2$

b $h(x) = -15$

$7 - 2x = -15$

$2x = 22$

$x = 11$

c $g(x) = h(x)$, so $4x - 5 = 7 - 2x$

$4x + 2x = 7 + 5$

$6x = 12$

$x = 2$

4 **a** $h(-3) = \frac{1}{-3-6} = -\frac{1}{9}$

b $x = 6$, as the denominator is zero and $h(x)$ is undefined.

5 **a** $f(5) = 5^3 = 125$

b The volume of a cube of side 5

6 **a** **i** $g(6) = \frac{3(6)+1}{(6)-2} = \frac{19}{4} = 4.75$

ii $g(-2) = \frac{3(-2)+1}{(-2)-2} = \frac{-5}{-4} = 1.25$

iii $g(0) = \frac{3(0)+1}{(0)-2} = \frac{1}{-2} = -0.5$

iv $g\left(-\frac{1}{3}\right) = \frac{3\left(-\frac{1}{3}\right)+1}{\left(-\frac{1}{3}\right)-2} = \frac{0}{-\frac{7}{3}} = 0$

b **i** $g(1) = \frac{3(1)+1}{(1)-2} = \frac{4}{-1} = -4$

ii $g(1.5) = \frac{3(1.5)+1}{(1.5)-2} = \frac{5.5}{-0.5} = -11$

iii $g(1.9) = \frac{3(1.9)+1}{(1.9)-2} = \frac{6.7}{-0.1} = -67$

iv $g(1.99) = \frac{3(1.99)+1}{(1.99)-2} = \frac{6.97}{-0.01} = -697$

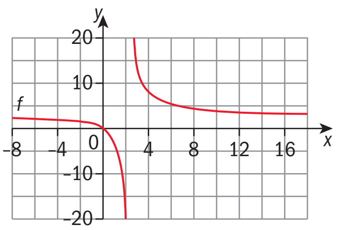
v $g(1.999) = \frac{3(1.999)+1}{(1.999)-2} = \frac{6.997}{-0.001} = -6997$

vi $g(1.9999) = \frac{3(1.9999)+1}{(1.9999)-2} = \frac{6.9997}{-0.0001} = -69997$

c The value of $g(x)$ is getting increasingly smaller as x approaches 2.

d 2 because the denominator equals zero when $x = 2$. Division by zero is undefined.

e



There is a vertical asymptote at $x = 2$, as $x = 2$ makes the denominator zero and $g(x)$ is undefined.

7 **a** The initial velocity occurs when $t = 0$.

$V(0) = (0^2 - 9)\ \text{ms}^{-1} = -9\ \text{ms}^{-1}$

b $V(4) = (4^2 - 9)\ \text{ms}^{-1} = 7\ \text{ms}^{-1}$

c $V(10) = (10^2 - 9)\ \text{ms}^{-1} = 91\ \text{ms}^{-1}$

d The particle comes to rest when $V(t) = 0$.

$t^2 - 9 = 0 \Rightarrow t^2 = 9 \Rightarrow t = 3\text{s}$.

8 **a** $f(2+h) = \frac{((2+h)+h)-(2+h)}{h} = \frac{2+h+h-2-h}{h} = \frac{h}{h} = 1$

b $f(3+h) = \frac{((3+h)+h)-(3+h))}{h} = \frac{3+h+h-3-h}{h} = \frac{h}{h} = 1$

Exercise 1F

1 **a** $(f \circ g)(3) = 3(3+1) = 12$

b $(f \circ g)(0) = 3(0+1) = 3$

c $(f \circ g)(-6) = 3(-6+1) = -15$

d $(f \circ g)(x) = 3(x+1) = 3x+3$

e $(g \circ f)(4) = (3(4))+1 = 13$

f $(g \circ f)(5) = (3(5))+1 = 16$

g $(g \circ f)(-6) = (3(-6))+1 = -17$

h $(g \circ f)(x) = (3(x))+1 = 3x+1$

i $(f \circ h)(2) = 3\left((2)^2+2\right) = 18$

j $(h \circ f)(2) = (3(2))^2 + 2 = 38$

k $(f \circ h)(x) = 3\left((x)^2+2\right) = 3x^2+6$

l $(h \circ f)(x) = (3(x))^2 + 2 = 9x^2 + 2$

m $(g \circ h)(3) = \left((3)^2+2\right)+1 = 12$

n $(h \circ g)(3) = (3+1)^2 + 2 = 18$

o $(g \circ h)(x) = \left((x)^2+2\right)+1 = x^2+3$

p $(h \circ g)(x) = (x+1)^2 + 2 = x^2 + 2x + 3$

2 **a** $(g \circ f)(1) = 3 - \left((1)^2 - 1\right) = 3$

b $(g \circ f)(2) = 3 - \left((2)^2 - 1\right) = 4 - 2^2 = 0$

c $(g \circ f)(4) = 3 - \left((4)^2 - 1\right) = -12$

d $(f \circ g)(3) = (3-(3))^2 - 1 = -1$

e $(g \circ f)(3) = 3 - \left((3)^2 - 1\right) = -5$

f $(f \circ g)(-4) = (3-(-4))^2 - 1 = 48$

g $(f \circ g)(x+1) = (3-(x+1))^2 - 1 = (2 - x)^2 - 1$

$= (4 - 2x + x^2) - 1 = 3 - 2x + x^2$

h $(f \circ g)(x+2) = (3-(x+2))^2 - 1 = (1 - x)^2 - 1$

$= (1 - 2x + x^2) - 1 = x^2 - 2x$

3 **a** $(f \circ g)(x) = (x+2)^2 = x^2 + 4x + 4$

b $(f \circ g)(3) = ((3)+2)^2 = 25$

4 **a** $(f \circ g)(x) = 5(x^2+1) = 5x^2 + 5$

b $(g \circ f)(x) = (5x)^2 + 1 = 25x^2 + 1$

5 **a** $(g \circ h)(x) = (x-4)^2 + 3 = x^2 - 8x + 19$

b $(h \circ g)(x) = (x^2+3) - 4 = x^2 - 1$

c $x^2 - 8x + 19 = x^2 - 1$

$-8x + 19 = -1$

$-8x = -20$

$x = 2.5$

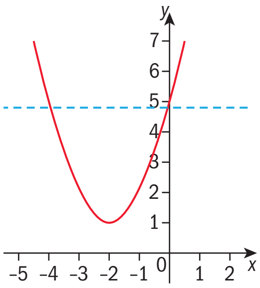
6 $(r \circ s)(x) = (x)^2 - 4 = x^2 - 4$

Domain $x \in \mathbb{R}$, Range $y \geq -4$

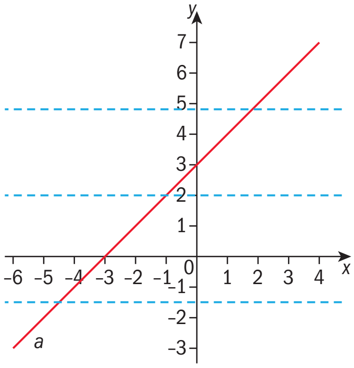
Exercise 1G

1 The following have inverse functions. **b**, **c**.

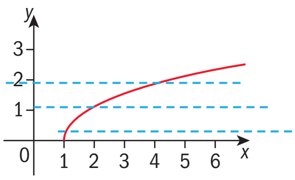
a No inverse function. Horizontal line crosses the graph twice.



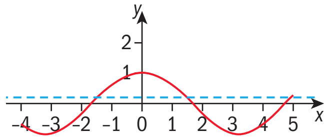
b Has an inverse function. Any horizontal line crosses the graph once only.



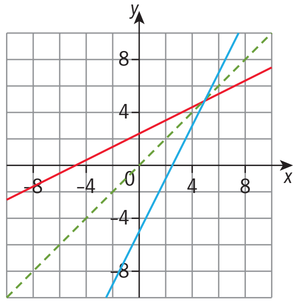
c Has an inverse function. Any horizontal line crosses the graph once only.



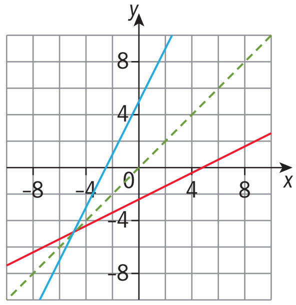
d No inverse function. Horizontal line crosses the graph more than once.



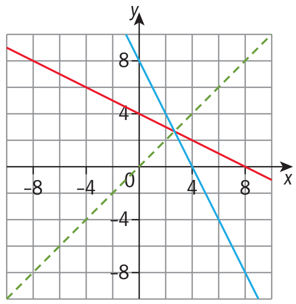
2 **a**



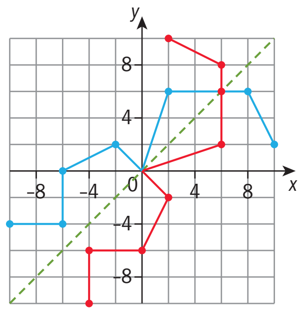
b



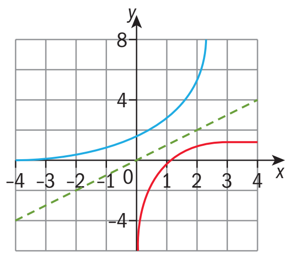
c



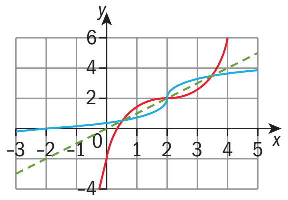
d



e



f



Exercise 1H

1 a i $g(1) = 2(1) - 4 = -2$ and

$$(f \circ g)(1) = \frac{(2(1)-4)+4}{2} = \frac{-2+4}{2} = 1$$

ii $f(-3) = \frac{-3+4}{2} = \frac{1}{2}$ and

$$(g \circ f)(-3) = 2\left(\frac{-3+4}{2}\right) - 4 = 2\left(\frac{1}{2}\right) - 4 = -3$$

iii $(f \circ g)(x) = \frac{(2(x)-4)+4}{2} = x$

iv $(g \circ f)(x) = 2\left(\frac{x+4}{2}\right) - 4 = x$

b they are inverses of each other

2 a
$x = 3y - 1$
$x + 1 = 3y$
$\frac{x+1}{3} = y$
$f^{-1}(x) = \frac{x+1}{3}$

b
$x = y^3 - 2$
$x + 2 = y^3$
$\sqrt[3]{x+2} = y$
$g^{-1}(x) = \sqrt[3]{x+2}$

c
$x = \frac{1}{4}y + 5$
$x - 5 = \frac{1}{4}y$
$4(x-5) = y$
$h^{-1}(x) = 4(x-5)$

d
$x = \sqrt[3]{y} - 3$
$x + 3 = \sqrt[3]{y}$
$(x+3)^3 = y$
$f^{-1}(x) = (x+3)^3$

e
$x = \frac{1}{y} - 2$
$x + 2 = \frac{1}{y}$
$y = \frac{1}{x+2}$
$g^{-1}(x) = \frac{1}{x+2}$

f
$x = 2y^3 + 3$
$x - 3 = 2y^3$
$\frac{x-3}{2} = y^3$
$\sqrt[3]{\frac{x-3}{2}} = y$
$h^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$

g
$x = \frac{y}{3+y}$
$x(3+y) = y$
$3x + xy = y$
$3x = y - xy$
$3x = y(1-x)$
$\frac{3x}{1-x} = y$
$f^{-1}(x) = \frac{3x}{1-x}$

h
$x = \frac{2y}{5-y}$
$x(5-y) = 2y$
$5x - xy = 2y$
$5x = 2y + xy$
$5x = y(2+x)$
$\frac{5x}{2+x} = y$
$f^{-1}(x) = \frac{5x}{x+2}$

3 a
$x = 1 - y$
$y + x = 1$
$y = 1 - x$
$f^{-1}(x) = 1 - x$

b
$x = y$
$f^{-1}(x) = x$

c
$x = \frac{1}{y}$
$xy = 1$
$y = \frac{1}{x}$
$f^{-1}(x) = \frac{1}{x}$

4 a
$f(x) = 6 - x$
$x = 6 - y$
$x - 6 = -y$
$6 - x = y$
$f^{-1}(x) = 6 - x$
$f^{-1}(5) = 6 - 5 = 1$

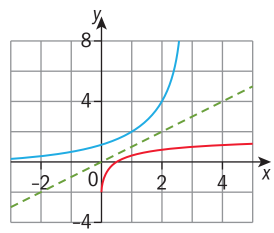
b
$f(x) = \frac{10}{x+7}$
$x = \frac{10}{y+7}$
$y + 7 = \frac{10}{x}$
$y = \frac{10}{x} - 7$
$f^{-1}(x) = \frac{10}{x} - 7$
$f^{-1}(5) = \frac{10}{5} - 7 = -5$

c
$f(x) = \frac{2}{4x-3}$
$x = \frac{2}{4y-3}$
$4y - 3 = \frac{2}{x}$
$4y = \frac{2}{x} + 3$
$y = \frac{1}{4}\left(\frac{2}{x} + 3\right)$
$f^{-1}(x) = \frac{1}{4}\left(\frac{2}{x} + 3\right)$
$f^{-1}(5) = \frac{1}{4}\left(\frac{2}{5} + 3\right) = \frac{1}{4} \times \frac{17}{5} = \frac{17}{20}$

5
$f(x) = \frac{x+1}{x-2}$
$x = \frac{y+1}{y-2}$
$x(y-2) = y + 1$
$xy - 2x = y + 1$
$xy - y = 2x + 1$
$y(x-1) = 2x + 1$
$y = \frac{2x+1}{x-1}$
$f^{-1}(x) = \frac{2x+1}{x-1}$

6

x	–3	–2	–1	0	1	2	3	4	5	6
f(x)	0.125	0.25	0.5	1	2	4	8	16	32	64



d $f(x)$. Domain $x \in \mathbb{R}$, Range $y > 0$

$f^{-1}(x)$. Domain $x > 0$, Range $y \in \mathbb{R}$.

7 $g(x) = \sqrt{x}$ has domain $x \geq 0$.

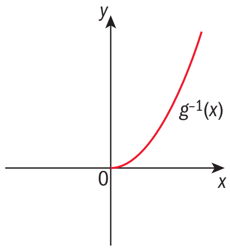
First, find $g^{-1}(x)$:

$x = \sqrt{y}$, $x \geq 0$

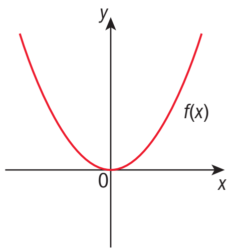
$x^2 = y$, $x \geq 0$

$g^{-1}(x) = x^2$, $x \geq 0$

The graph of $g^{-1}(x)$ is shown below.



You can see that $g^{-1}(x)$ has domain $x \geq 0$
range $g^{-1}(x) \geq 0$.
Now, the graph of $f(x) = x^2$ is shown below:



You can see that $f(x)$ has domain $x \in \mathbb{R}$
range $f(x) \geq 0$.
Hence, $f(x)$ and $g^{-1}(x)$ are different.

8 Let $f(x) = mx + c$.

$x = my + c$

$x - c = my$

$\frac{x}{m} - \frac{c}{m} = y$

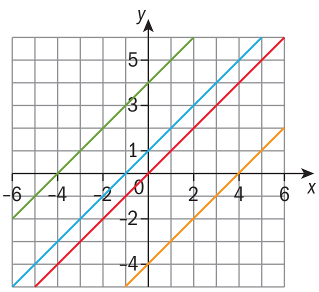
$f^{-1}(x) = \frac{1}{m}x - \frac{c}{m}$

For graphs of *$f(x)$ and $f^{-1}(x)$ to be perpendicular,*

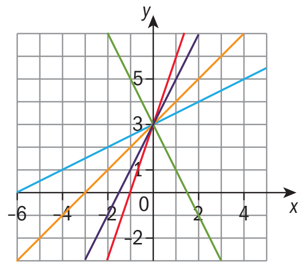
$m \times \frac{1}{m}$ *should be –1 but* $m \times \frac{1}{m} = 1$.

Investigation – functions

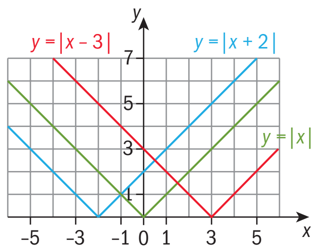
1 Changing the constant term translates $y = x$ along the y-axis.



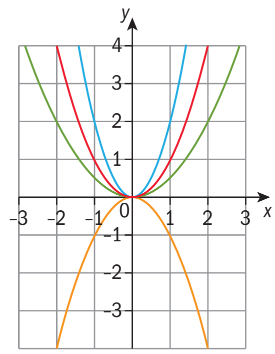
2 Changing the x-coefficient alters the gradient of the line.



3 $y = |x + h|$ is a translation of $-h$ along the x-axis

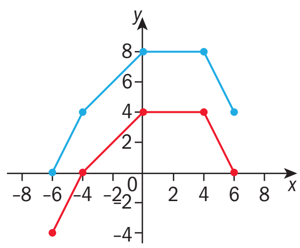


4 The negative sign reflects the graph in the x-axis. Increasing the value of a means the graph increases more steeply.

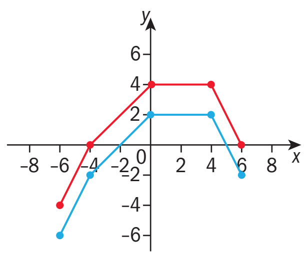


Exercise 1I

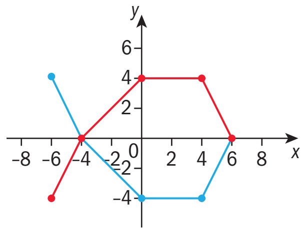
1 a



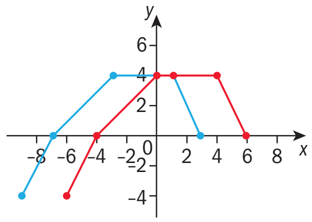
b



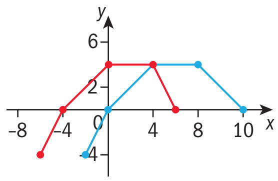
c



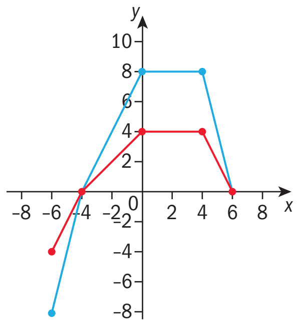
d



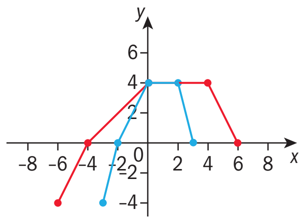
e



f



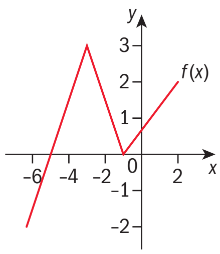
g



2 g is a vertical translation of 2 units, so $g(x) = f(x) + 2$.
h is is a vertical translation of -4 units, so $h(x) = f(x) - 4$.
q is a horizontal stretch of scale factor 2, so $q(x) = f(\frac{1}{2}x)$.

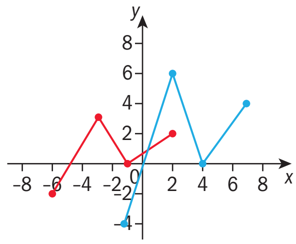
3 q is a horizontal translation of -4 and a vertical translation of -2, so $q(x) = f(x + 4) - 2$
s is a horizontal translation of -4, so $s(x) = f(x + 4)$.
t is a horizontal translation of 2, so $t(x) = f(x - 2)$.

4



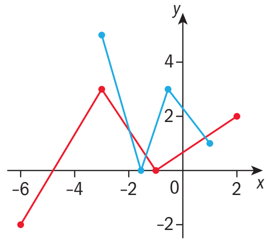
This is the graph of $f(x)$. It has domain $-6 \le x \le 2$
It has range $-2 \le f(x) \le 3$

a $2f(x - 5)$ is a horizontal translation of 5, followed by a vertical stretch of scale factor 2.



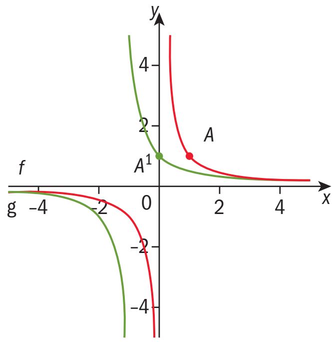
$2f(x - 5)$ has domain $-1 \le x \le 7$, and has range $-4 \le 2f(x - 5) \le 6$.

b $-f(2x) + 3$ is a horizontal stretch of scale factor $\frac{1}{2}$ followed by reflection in the x-axis, followed by a vertical translation of 3.

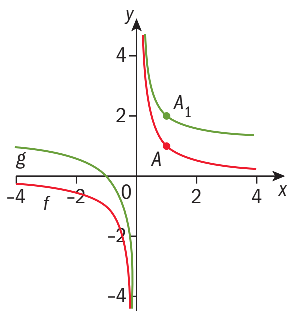


$-f(2x) + 3$ has domain $-3 \le x \le 1$, and has range $0 \le y \le 5$.

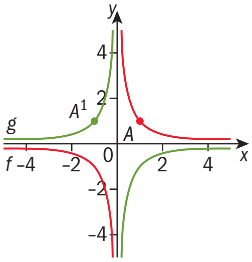
5 **a** $f(x + 1)$ is a horizontal translation of $f(x)$ by -1 units.



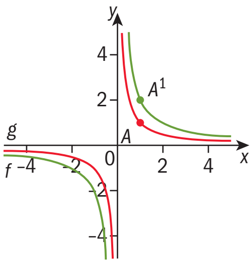
b $f(x) + 1$ is a vertical translation of $f(x)$ by $+1$ unit.



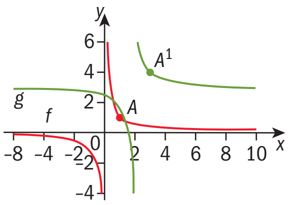
c $f(-x)$ is a reflection of $f(x)$ in the y-axis.



d $2f(x)$ is a vertical stretch of $f(x)$ by scale factor 2.



e $f(x-2)+3$ is a horizontal translation of $f(x)$ by 2 units, followed by a vertical translation of 3 units.

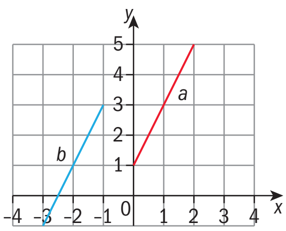


6 **a** Reflection in the x-axis.

b Horizontal translation of 3 units.

c A vertical stretch of scale factor 2 followed by a reflection in the x-axis and then a vertical translation of 5 units.

7



$g(x)$ is a horizontal translation of $f(x)$ by -3 units, followed by a vertical translation of -2 units.

Review exercise

1 **a** $g(a-2) = 4(a-2) - 5 = 4a - 8 - 5$
$= 4a - 13.$

b $h(1-x)=\frac{1+(1-x)}{1-(1-x)}=\frac{2-x}{x}$

2 **a** $f(x-3) = 2(x-3)^2 - 3(x-3) + 1$
$= 2x^2 - 12x + 18 - 3x + 9 + 1$
$= 2x^2 - 15x + 28$

b

$$(f\circ g)(x)=2(1-x^2)+7=2-2x^2+7=-2x^2+9$$

3 **a** $f(x)=\frac{3x+17}{2}$

$x=\frac{3y+17}{2}$

$2x=3y+17$

$2x-17=3y$

$\frac{2x-17}{3}=y$

$f^{-1}(x)=\frac{2x-17}{3}$

b $g(x)=5x^3-4$

$x=5y^3-4$

$x+4=5y^3$

$\frac{x+4}{5}=y^3$

$\sqrt[3]{\frac{x+4}{5}}=y$

$g^{-1}(x)=\sqrt[3]{\frac{x+4}{5}}$

4 $f(x)=-\frac{1}{5}x-1$

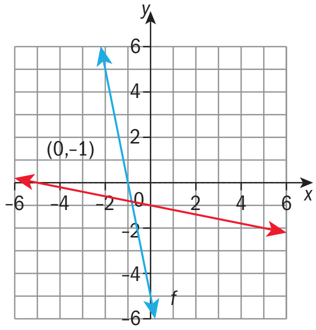
$x=-\frac{1}{5}y-1$

$x+1=-\frac{1}{5}y$

$-5(x+1)=y$

$-5x-5=y$

$f^{-1}(x)=-5x-5$



5 **a** $f(x)=3x+5$

$x=3y+5$

$x-5=3y$

$\frac{x-5}{3}=y$

$f^{-1}(x)=\frac{x-5}{3}$

b $f(x)=\sqrt[3]{x+2}$

$x=\sqrt[3]{y+2}$

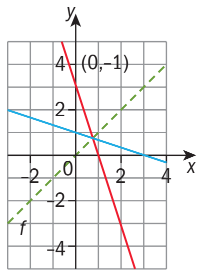
$x^3=y+2$

$x^3-2=y$

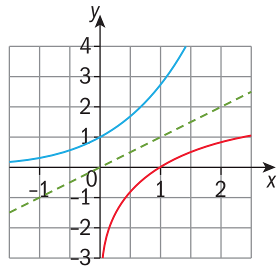
$f^{-1}(x)=x^3-2$

6 Reflect each graph in the line $y = x$

a



b



7 **a** Domain $x\in\mathbb{R}$, Range $y\geq 0$

b Domain $x\in\mathbb{R}$, $x\neq 3$, Range $y\in\mathbb{R}$, $y\neq 0$

8 **a** Reflect in the y-axis. $f(x) = -x$

Vertical stretch scale factor 2. $f(x) = -2x$

Horizontal stretch scale factor $\frac{1}{3}$. $f(x) = -2(3x)$

Translate 3 units left. $f(x) = -2(3x + 3)$

Translate 2 units up. $f(x) = -2(3x + 3) + 2$

Expand and simplify. $f(x) = -6x - 4 = -2(3x + 2)$

b Reflect in the x axis. $f(x) = -(x^2)$

Stretch vertically by scale factor $\frac{1}{4}$.

$f(x) = -\frac{1}{4}(x)^2$

Stretch horizontally scale factor 3.

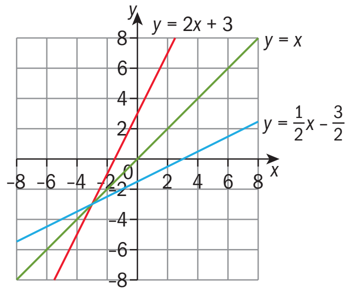
$f(x) = -\frac{1}{4}\left(\frac{1}{3}x\right)^2$

Translate 5 units right. $f(x) = -\frac{1}{4}\left(\frac{1}{3}x - 5\right)^2$

Translate 1 unit down. $f(x) = -\frac{1}{4}\left(\frac{1}{3}x - 5\right)^2 - 1$

9 a The graph of an inverse function is the reflection of the graph of the original function in the line $y = x$.

b Graph a line with a y-intercept of 3 and slope of 2. Draw the line $y = x$. To graph its inverse, sketch the mirror image of the original line.



10 a $g(0) = 3(0) - 2 = -2$

b $(f \circ g)(0) = 2(-2)^3 + 3 = -16 + 3 = -13$

c $f(x) = 2x^3 + 3$

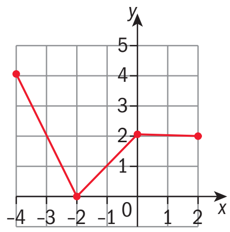
$x = 2y^3 + 3$

$x - 3 = 2y^3$

$\frac{x-3}{2} = y^3$

$f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$

11 a $f(-x)$ is a reflection of $f(x)$ in the line $x = 0$.



b $g(x) = \frac{1}{2}f(x-1)$ describes the transformation: Horizontal translation by 1 unit, followed by vertical stretch, scale factor $\frac{1}{2}$. so P is (4, 1)

12 a $(f \circ g)(x) = 3(x + 2) = 3x + 6$

b $f^{-1}(x) = \frac{x}{3}$ and $g^{-1}(x) = x - 2$

$f^{-1}(12) = \frac{12}{3} = 4$

$g^{-1}(12) = 12 - 2 = 10$

$f^{-1}(12) + g^{-1}(12) = 4 + 10$

$f^{-1}(12) + g^{-1}(12) = 14$

13 a $(h \circ g)(x) = \frac{3(2x-1)}{(2x-1)-2} = \frac{6x-3}{2x-3}$

b $\frac{6x-3}{2x-3} = 0$

$6x - 3 = 0$

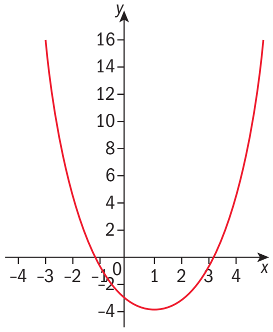
$6x = 3$

$x = \frac{1}{2}$

Review exercise

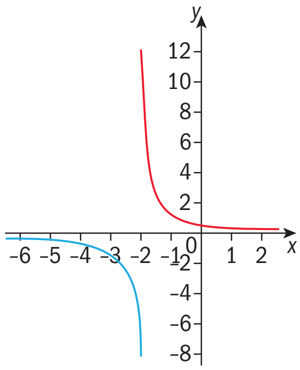
1 Domain: $x \geq -2$ Range: $y > 0$

2



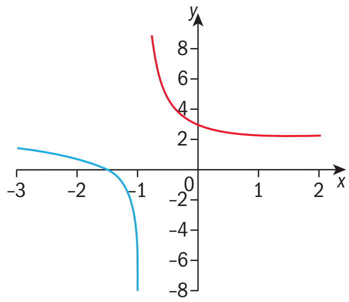
Domain: $x \in \mathbb{R}$ Range: $y \geq -4$

3



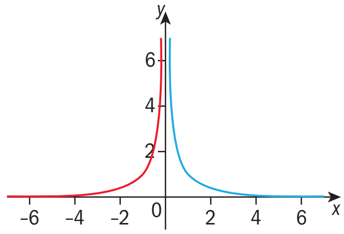
Domain: $x \in \mathbb{R}$, $x \neq -2$ Range: $y \in \mathbb{R}$, $y \neq 0$

4 a



b x-intercept -1.5, y-intercept 3.

5 a

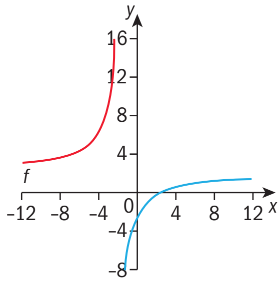


b 0

c Domain: $x \in \mathbb{R}$, $x \neq 0$. Range: $y > 0$.

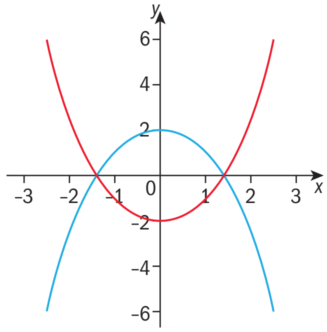
6 **a** $x = -2,\ y = 2.$

b



c $(2.5, 0), (0, -2.5)$

7 **a**



b $\pm\sqrt{2}$

8 **a** $f(x) = x^3 - 3$

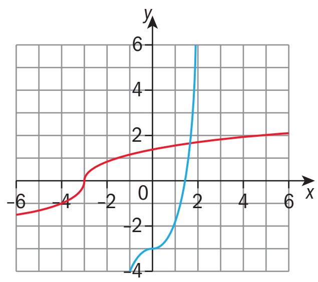
$x = y^3 - 3$

$x + 3 = y^3$

$y = \sqrt[3]{x+3}$

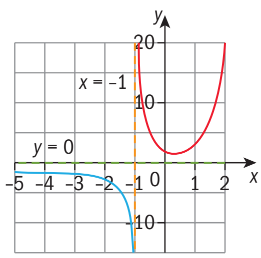
$f^{-1}(x) = \sqrt[3]{x+3}$

b



c 1.67

9



10 **a** $f(x) = 3x - 2$

$x = 3y - 2$

$x + 2 = 3y$

$y = \dfrac{x+2}{3}$

$f^{-1}(x) = \dfrac{x+2}{3}$

b $(g^{-1} \circ f)(x) = (3x - 2) + 3 = 3x + 1$

c $\left(f^{-1} \circ g\right) = \frac{(x-3)+2}{3} = \frac{x-1}{3}$

d $\left(f^{-1} \circ g\right)(x) = \left(g^{-1} \circ f\right)(x)$, so

$\frac{x-1}{3} = 3x + 1$

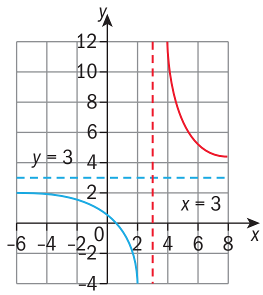
$x - 1 = 3(3x+1)$

$x - 1 = 9x + 3$

$8x = -4$

$x = -\frac{1}{2}$

e



f $x = 3,\ y = 3$

2 Quadratic functions and equations

Answers

Skills check

1 **a** $3a-5=a+7$
$2a = 12$
$a = 6$

b $4x^2+1=21$
$4x^2 = 20$
$x^2 = 5$
$x=\pm\sqrt{5}$

c $3(n-4) = 5(n+2)$
$3n - 12 = 5n + 10$
$2n = -22$
$n = -11$

2 **a** $2k(k-5)$

b $7a(2a^2 + 3a - 7)$

c $2x^2+4xy+3x+6y$
$2x(x+2y)+3(x+2y)$
$(2x+3)(x+2y)$

d $5a^2-10a-ab+2b$
$5a(a-2)-b(a-2)$
$(5a-b)(a-2)$

e $(n+1)(n+3)$

f $(2x-3)(x+1)$

g $(m+6)(m-6)$

h $(5x+9y)(5x-9y)$

Exercise 2A

1 **a** $x^2 - 3x + 2 = 0$
$x^2 - x - 2x + 2 = 0$
$x(x-1) - 2(x-1) = 0$
$(x-1)(x-2) = 0$
$x = 1$ or $x = 2$

b $a^2 + a - 56 = 0$
$a^2 + 8a - 7a - 56 = 0$
$a(a+8) - 7(a+8) = 0$
$(a+8)(a-7) = 0$
$a = -8$ or $a = 7$

c $m^2 - 11m + 30 = 0$
$m^2 - 5m - 6m + 30 = 0$
$m(m-5) - 6(m-5) = 0$
$(m-5)(m-6) = 0$
$m = 5$ or $m = 6$

d $x^2 - 25 = 0$
$(x+5)(x-5) = 0$
$x = \pm 5$

e $x^2 + 2x - 48 = 0$
$x^2 + 8x - 6x - 48 = 0$
$x(x+8) - 6(x+8) = 0$
$(x+8)(x-6) = 0$
$x = -8$ or $x = 6$

f $b^2 + 6b + 9 = 0$
$b^2 + 3b + 3b + 9 = 0$
$b(b+3) + 3(b+3) = 0$
$(b+3)^2 = 0$
$b = -3$

2 **a** $6x^2 + 5x - 4 = 0$
$6x^2 + 8x - 3x - 4 = 0$
$2x(3x+4) - 1(3x+4) = 0$
$(3x+4)(2x-1) = 0$
$x=-\frac{4}{3}$ or $x=\frac{1}{2}$

b $5c^2 + 6c - 8 = 0$
$5c^2 + 10c - 4c - 8 = 0$
$5c(c+2) - 4(c+2) = 0$
$(c+2)(5c-4) = 0$
$c = -2$ or $c=\frac{4}{5}$

c $2h^2 - 3h - 5 = 0$
$2h^2 - 5h + 2h - 5 = 0$
$h(2h-5) + 1(2h-5) = 0$
$(h+1)(2h-5) = 0$
$h = -1$ or $h=\frac{5}{2}$

d $4x^2 - 16x - 9 = 0$
$4x^2 - 18x + 2x - 9 = 0$
$2x(2x-9) + 1(2x-9) = 0$
$(2x+1)(2x-9) = 0$
$x=-\frac{1}{2}$ or $x=\frac{9}{2}$

e $3t^2 + 14t + 8 = 0$
$3t^2 + 12t + 2t + 8 = 0$
$3t(t+4) + 2(t+4) = 0$
$(t+4)(3t+2) = 0$
$t = -4$ or $t=-\frac{2}{3}$

f $6x^2 + x - 12 = 0$

$6x^2 + 9x - 8x - 12 = 0$

$3x(2x + 3) - 4(2x + 3) = 0$

$(2x + 3)(3x - 4) = 0$

$x = -\frac{3}{2}$ or $x = \frac{4}{3}$

Exercise 2B

1 a $x^2 + 2x - 7 = 13 + x$

$x^2 + x - 20 = 0$

$x^2 + 5x - 4x - 20 = 0$

$x(x + 5) - 4(x + 5) = 0$

$(x + 5)(x - 4) = 0$

$x = -5$ or $x = 4$

b $2n^2 + 11n = 3n - n^2 - 4$

$3n^2 + 8n + 4 = 0$

$3n^2 + 6n + 2n + 4 = 0$

$3n(n + 2) + 2(n + 2) = 0$

$(n + 2)(3n + 2) = 0$

$n = -2$ or $n = -\frac{2}{3}$

c $3z^2 + 12z = -z^2 - 9$

$4z^2 + 12z + 9 = 0$

$4z^2 + 6z + 6z + 9 = 0$

$2z(2z + 3) + 3(2z + 3) = 0$

$(2z + 3)(2z + 3) = 0$

$z = -\frac{3}{2}$

d $2a^2 - 50 = 21a$

$2a^2 - 21a - 50 = 0$

$2a^2 - 25a + 4a - 50 = 0$

$a(2a - 25) + 2(2a - 25) = 0$

$(a + 2)(2a - 25) = 0$

$a = -2$ or $a = \frac{25}{2}$

e $x^2 + 5x = 36$

$x^2 + 5x - 36 = 0$

$x^2 + 9x - 4x - 36 = 0$

$x(x + 9) - 4(x + 9) = 0$

$(x + 9)(x - 4) = 0$

$x = -9$ or $x = 4$

f $4x^2 - 2x = x + 1$

$4x^2 - 3x - 1 = 0$

$4x^2 - 4x + x - 1 = 0$

$4x(x - 1) + 1(x - 1) = 0$

$(4x + 1)(x - 1) = 0$

$x = -\frac{1}{4}$ or $x = 1$

2 $x^2 - x = 12$

$x^2 - x - 12 = 0$

$x^2 - 4x + 3x - 12 = 0$

$x(x - 4) + 3(x - 4) = 0$

$(x + 3)(x - 4) = 0$

$x = -3$ or $x = 4$

3 $(x + 2)^2 + (5x - 3)^2 = (4x + 1)^2$

$x^2 + 4x + 4 + 25x^2 - 30x + 9 = 16x^2 + 8x + 1$

$10x^2 - 34x + 12 = 0$

$2(5x^2 - 17x + 6) = 0$

$2(5x^2 - 15x - 2x + 6) = 0$

$2(5x(x - 3) - 2(x - 3) = 0$

$2(5x - 2)(x - 3) = 0$

$x = 3$

If $x = \frac{2}{5}$, the leg $5x - 3$ would have a negative length. Therefore, the only answer is $x = 3$.

Investigation – perfect square trinomials

1 $x^2 + 10x + 25 = 0 \rightarrow (x + 5)(x + 5) = (x + 5)^2 = 0$

$x = -5$

2 $x^2 + 6x + 9 = 0 \rightarrow (x + 3)(x + 3) = (x + 3)^2 = 0$

$x = -3$

3 $x^2 + 14x + 49 = 0 \rightarrow (x + 7)(x + 7) = (x + 7)^2 = 0$

$x = -7$

4 $x^2 - 8x + 16 = 0 \rightarrow (x - 4)(x - 4) = (x - 4)^2 = 0$

$x = 4$

5 $x^2 - 18x + 81 = 0 \rightarrow (x - 9)(x - 9) = (x - 9)^2 = 0$

$x = 9$

6 $x^2 - 20x + 100 = 0 \rightarrow (x - 10)(x - 10)$

$= (x - 10)^2 = 0$

$x = 10$

Exercise 2C

1 $x^2 + 8x + 16 = 3 + 16$

$(x + 4)^2 = 19$

$x + 4 = \pm\sqrt{19}$

$x = -4 \pm \sqrt{19}$

2 $x^2 - 5x + \frac{25}{4} = 3 + \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 = \frac{37}{4}$

$x - \frac{5}{2} = \pm\sqrt{\frac{37}{4}} = \frac{\pm\sqrt{37}}{2}$

$x = \frac{5 \pm \sqrt{37}}{2}$

3 $x^2 - 6x = -1$

$x^2 - 6x + 9 = -1 + 9$

$(x - 3)^2 = 8$

$x - 3 = \pm\sqrt{8} = \pm 2\sqrt{2}$

$x = 3 \pm 2\sqrt{2}$

4 $x^2 + 7x = 4$

$x^2 + 7x + \frac{49}{4} = 4 + \frac{49}{4}$

$\left(x + \frac{7}{2}\right)^2 = \frac{65}{4}$

$x + \frac{7}{2} = \pm\sqrt{\frac{65}{4}} = \frac{\pm\sqrt{65}}{2}$

$x = \frac{-7 \pm \sqrt{65}}{2}$

5 $x^2 - 2x = 6$

$x^2 - 2x + 1 = 6 + 1$

$(x - 1)^2 = 7$

$x - 1 = \pm\sqrt{7}$

$x = 1 \pm \sqrt{7}$

6 $x^2 + x = 3$

$x^2 + x + \frac{1}{4} = 3 + \frac{1}{4}$

$\left(x + \frac{1}{2}\right)^2 = \frac{13}{4}$

$x + \frac{1}{2} = \pm\sqrt{\frac{13}{4}} = \frac{\pm\sqrt{13}}{2}$

$x = \frac{-1 \pm \sqrt{13}}{2}$

Exercise 2D

1 $x^2 + 6x = 3$

$x^2 + 6x + 9 = 3 + 9$

$(x + 3)^2 = 12$

$x + 3 = \pm\sqrt{12} = \pm 2\sqrt{3}$

$x = -3 \pm 2\sqrt{3}$

2 $x^2 - 2x = 1$

$x^2 - 2x + 1 = 1 + 1$

$(x - 1)^2 = 2$

$x - 1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$

3 $5\left(x^2 - 2x\right) = -2$

$x^2 - 2x = -\frac{2}{5}$

$x^2 - 2x + 1 = -\frac{2}{5} + 1$

$(x - 1)^2 = \frac{3}{5}$

$x - 1 = \pm\sqrt{\frac{3}{5}}$

$x = 1 \pm \sqrt{\frac{3}{5}}$

4 $4\left(x^2 + \frac{3}{2}x\right) = 5$

$x^2 + \frac{3}{2}x = \frac{5}{4}$

$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{5}{4} + \frac{9}{16}$

$\left(x + \frac{3}{4}\right)^2 = \frac{29}{16}$

$x + \frac{3}{4} = \pm\sqrt{\frac{29}{16}} = \frac{\pm\sqrt{29}}{4}$

$x = \frac{-3 \pm \sqrt{29}}{4}$

5 $2\left(x^2 - \frac{1}{2}x\right) = 6$

$x^2 - \frac{1}{2}x = 3$

$x^2 - \frac{1}{2}x + \frac{1}{16} = 3 + \frac{1}{16}$

$\left(x - \frac{1}{4}\right)^2 = \frac{49}{16}$

$x - \frac{1}{4} = \pm\sqrt{\frac{49}{16}} = \pm\frac{7}{4}$

$x = \frac{1}{4} \pm \frac{7}{4}$

$x = -\frac{3}{2}, 2$

6 $10\left(x^2 + \frac{2}{5}x\right) = 5$

$x^2 + \frac{2}{5}x = \frac{1}{2}$

$x^2 + \frac{2}{5}x + \frac{1}{25} = \frac{1}{2} + \frac{1}{25}$

$\left(x + \frac{1}{5}\right)^2 = \frac{27}{50}$

$x + \frac{1}{5} = \pm\sqrt{\frac{27}{50}} = \pm\frac{3\sqrt{3}}{5\sqrt{2}} = \frac{\pm 3\sqrt{6}}{10}$

$x = \frac{-2 \pm 3\sqrt{6}}{10}$

Exercise 2E

1 $x = \frac{-9 \pm \sqrt{(9)^2 - 4(4)(-7)}}{2(4)} = \frac{-9 \pm \sqrt{81 + 112}}{8}$

$x = \frac{-9 \pm \sqrt{193}}{8}$

2 $x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(-8)}}{2(3)} = \frac{-2 \pm \sqrt{4 + 96}}{6} = \frac{-2 \pm \sqrt{100}}{6}$

$x = \frac{-2 \pm 10}{6}$

$x = -2, \frac{4}{3}$

3 $x = \frac{-6 \pm \sqrt{(6)^2 - 4(5)(1)}}{2(5)} = \frac{-6 \pm \sqrt{36 - 20}}{10} = \frac{-6 \pm \sqrt{16}}{10}$

$x = \frac{-6 \pm 4}{10}$

$x = -1, -\frac{1}{5}$

4 $x^2 - 6x + 4 = 0$

$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = \frac{6 \pm \sqrt{36 - 16}}{2} = \frac{6 \pm \sqrt{20}}{2}$

$x = \frac{6 \pm 2\sqrt{5}}{2} = 3 \pm \sqrt{5}$

5 $x^2 - x + 3 = 0$

$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(3)}}{2(1)} = \frac{1 \pm \sqrt{1 - 12}}{2} = \frac{1 \pm \sqrt{-11}}{2}$

no solution since $\sqrt{-11}$ is undefined.

6 $3x^2 + 10x - 5 = 0$

$x = \frac{-10 \pm \sqrt{(10)^2 - 4(3)(-5)}}{2(3)} = \frac{-10 \pm \sqrt{100 + 60}}{6} = \frac{-10 \pm \sqrt{160}}{6}$

$x = \frac{-10 \pm 4\sqrt{10}}{6} = \frac{-5 \pm 2\sqrt{10}}{3}$

7 $2x^2 - 3x - 1 = 0$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} = \frac{3 \pm \sqrt{9 + 8}}{4}$

$x = \frac{3 \pm \sqrt{17}}{4}$

8 $2x^2 - 9x - 4 = 0$

$$x = \frac{9 \pm \sqrt{(-9)^2 - 4(2)(-4)}}{2(2)} = \frac{9 \pm \sqrt{81 + 32}}{4}$$

$$x = \frac{9 \pm \sqrt{113}}{4}$$

9 $6 - 2x^2 = 9x$

$2x^2 + 9x - 6 = 0$

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4(2)(-6)}}{2(2)} = \frac{-9 \pm \sqrt{81 + 48}}{4}$$

$$x = \frac{-9 \pm \sqrt{129}}{4}$$

10 $(5x-2)(x) = (x+3)(x+1)$

$5x^2 - 2x = x^2 + 4x + 3$

$4x^2 - 6x - 3 = 0$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(4)(-3)}}{2(4)} = \frac{6 \pm \sqrt{36+48}}{8}$$

$$x = \frac{6 \pm \sqrt{84}}{8} = \frac{6 \pm 2\sqrt{21}}{8} = \frac{3 \pm \sqrt{21}}{4}$$

Exercise 2F

1 Let the two numbers be x and y.

$x + y = 50 \rightarrow y = 50 - x$

$xy = 576 \rightarrow x(50 - x) = 576$

$50x - x^2 = 576 \rightarrow x^2 - 50x^2 + 576 = 0$

$(x - 18)(x - 32) = 0$

The possible values for x are 18 and 32.
Since $y = 50 - x$, the two numbers are 18 and 32.

> This quadratic equation could also be solved using completing the square or the quadratic formula.

2 Let l represent the length, and w represent the width.

$2l + 2w = 70 \rightarrow l + w = 35 \rightarrow l = 35 - w$

$lw = 264 \rightarrow (35 - w)w = 264$

$35w - w^2 = 264 \rightarrow w^2 - 35w + 264 = 0$

$(w - 11)(w - 24) = 0$

The dimensions are 24 m and 11 m.

3 $(x + 6)^2 + (3x)^2 = (4x - 6)^2$

$x^2 + 12x + 36 + 9x^2 = 16x^2 - 48x + 36$

$6x^2 - 60x = 0$

$6x(x - 10) = 0$

$x = 10$ (we cannot have $x = 0$, since this would mean one side has zero length).

4 $(23 - x)(16 + x) = 378$

$368 + 7x - x^2 = 378 \rightarrow x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

$x = 2, 5$

The dimensions are 18 cm and 21 cm.

5 $h = 2 + 14t - 4.9t^2$

The ball hits the ground when $h = 0$.

$0 = 2 + 14t - 4.9t^2$

$$t = \frac{-14 \pm \sqrt{(14)^2 - 4(-4.9)(2)}}{2(-4.9)} = \frac{-14 \pm \sqrt{235.2}}{-9.8}$$

approximately 2.99 seconds. We cannot have 'negative time', so we take only the positive value.

Investigation – roots of quadratic equations

1 **a** $x = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} = \dfrac{8 \pm \sqrt{0}}{2} = \dfrac{8}{2} = 4$

b $x = \dfrac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} = \dfrac{12 \pm \sqrt{0}}{8} = \dfrac{12}{8} = \dfrac{3}{2}$

c $x = \dfrac{-10 \pm \sqrt{(10)^2 - 4(25)(1)}}{2(25)} = \dfrac{-10 \pm \sqrt{0}}{50} = \dfrac{-10}{50} = -\dfrac{1}{5}$

2 **a** $x = \dfrac{-5 \pm \sqrt{(5)^2 - 4(1)(-14)}}{2(1)} = \dfrac{-5 \pm \sqrt{81}}{2} = \dfrac{-5 \pm 9}{2}$

$x = -7, 2$

b $x = \dfrac{8 \pm \sqrt{(-8)^2 - 4(3)(2)}}{2(3)} = \dfrac{8 \pm \sqrt{40}}{6} = \dfrac{8 \pm 2\sqrt{10}}{6}$

$x = \dfrac{4 \pm \sqrt{10}}{3}$

c $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4(5)(-4)}}{2(5)} = \dfrac{3 \pm \sqrt{89}}{10}$

3 **a** $x = \dfrac{-3 \pm \sqrt{(3)^2 - 4(1)(6)}}{2(1)} = \dfrac{-3 \pm \sqrt{-15}}{2}$

no solution

b $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)} = \dfrac{4 \pm \sqrt{-24}}{4}$

no solution

c $x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(4)(1)}}{2(4)} = \dfrac{-2 \pm \sqrt{-12}}{8}$

no solution

Exercise 2G

1 **a** $\Delta = 5^2 - 4(1)(-3) = 37$

two different real roots

b $\Delta = 4^2 - 4(2)(1) = 8$

two different real roots

c $\Delta = (-1)^2 - 4(4)(5) = -79$

no real roots

d $\Delta = (8)^2 - 4(1)(16) = 0$

two equal real roots

e $\Delta = (-3)^2 - 4(1)(8) = -23$

no real roots

f $\Delta = (-20)^2 - 4(12)(25) = -800$

no real roots

2 **a** $4^2 - 4(1)(p) > 0$

$16 > 4p$

$p < 4$

b $5^2 - 4(p)(2) > 0$

$25 > 8p$

$p < \dfrac{25}{8}$

c $p^2 - 4(1)(8) > 0$

$p^2 > 32 \Rightarrow p < -\sqrt{32}$ *or* $p > \sqrt{32}$, so

$|p| > \sqrt{32}$

$|p| > 4\sqrt{2}$

d $(3p)^2 - 4(1)(1) > 0$

$9p^2 > 4$

$p^2 > \frac{4}{9} \Rightarrow p < \frac{-2}{3}$ or $p > \frac{2}{3}$, so

$|p| > \frac{2}{3}$

3 **a** $10^2 - 4(1)(k) = 0$

$100 = 4k$

$k = 25$

b $(-3)^2 - 4(2)(k) = 0$

$9 = 8k$

$k = \frac{9}{8}$

c $(-2k)^2 - 4(3)(5) = 0$

$4k^2 = 60$

$k^2 = 15$

$k = \pm\sqrt{15}$

d $(-4k)^2 - 4(1)(-3k) = 0$

$16k^2 + 12k) = 0$

$4k(4k + 3) = 0$

$k = 0, -\frac{3}{4}$

4 **a** $(-6)^2 - 4(1)(m) < 0$

$36 < 4m$

$m > 9$

b $(5m)^2 - 4(1)(25) < 0$

$25m^2 < 100$

$m^2 < 4$

$-2 < m < 2$

c $(-8)^2 - 4(3m)(1) < 0$

$64 < 12m$

$m > \frac{16}{3}$

d $(6)^2 - 4(1)(m - 3) < 0$

$36 < 4m - 12$

$48 < 4m$

$m > 12$

5 $(-4q)^2 - 4(q)(5 - q) < 0$

$16q^2 - 20q + 4q^2 < 0$

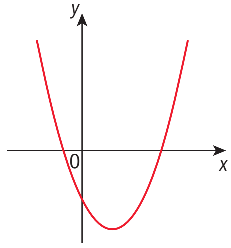
$20q^2 - 20q < 0$

$20q(q - 1) < 0$

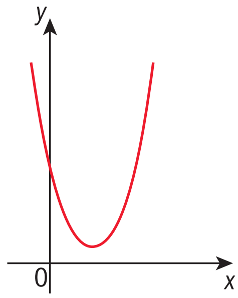
$0 < q < 1$

Investigation – graphs of quadratic functions

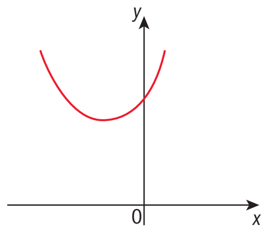
a $\Delta = (-3)^2 - 4(1)(-5) = 29$



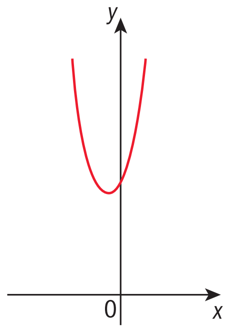
b $\Delta = (-6)^2 - 4(3)(4) = -12$



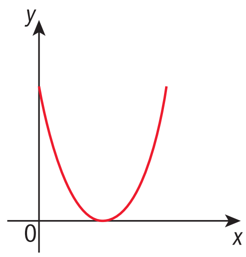
c $\Delta = (2)^2 - 4(1)(7) = -24$



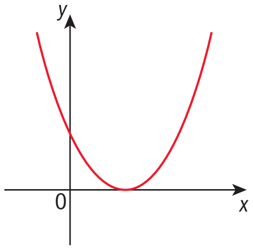
d $\Delta = (3)^2 - 4(4)(5) = -71$



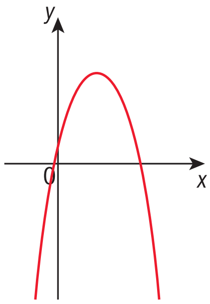
e $\Delta = (-6)^2 - 4(1)(9) = 0$



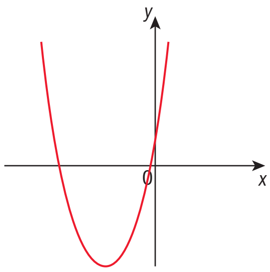
f $\Delta = (-4)^2 - 4(2)(2) = 0$



g $\Delta = (5)^2 - 4(-1)(2) = 33$



h $\Delta = (7)^2 - 4(1)(3) = 37$



Exercise 2H

1 **a** $x = \frac{-8}{2(1)} = -4$; $(0, 5)$

b $x = \frac{6}{2(1)} = 3$; $(0, -3)$

c $x = \frac{-10}{2(5)} = -1$; $(0, 6)$

d $x = \frac{-10}{2(-3)} = \frac{-10}{-6} = \frac{5}{3}$; $(0, 9)$

2 **a** vertex $(7, -2)$

$y = (x-7)^2 - 2 = x^2 - 14x + 47 \rightarrow$ y-intercept $(0, 47)$

b vertex $(-5, 1)$

$y = (x+5)^2 + 1 = x^2 + 10x + 26 \rightarrow$ y-intercept $(0, 26)$

c vertex $(1, 6)$

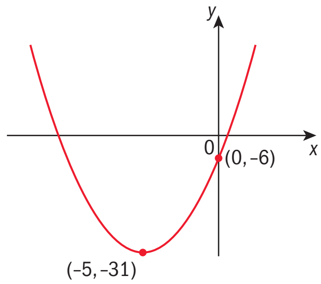
$y = 4(x-1)^2 + 6 = 4x^2 - 8x + 10 \rightarrow$ y-intercept $(0, 10)$

d vertex $(-2, -7)$

$y = 3(x+2)^2 - 7 = 3x^2 + 12x + 5 \rightarrow$ y-intercept $(0, 5)$

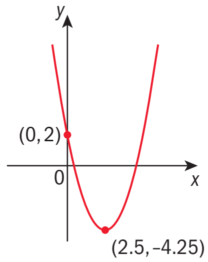
3 **a** $f(x) = x^2 + 10x - 6 = (x^2 + 10x + 25) - 6 - 25$

$f(x) = (x+5)^2 - 31$



b $f(x) = x^2 - 5x + 2 = (x^2 - 5x + 6.25) + 2 - 6.25$

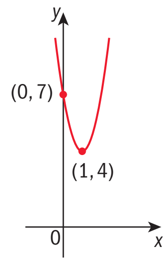
$f(x) = (x - 2.5)^2 - 4.25$



c $f(x) = 3x^2 - 6x + 7 = 3(x^2 - 2x) + 7$

$= 3(x^2 - 2x + 1) + 7 - 3$

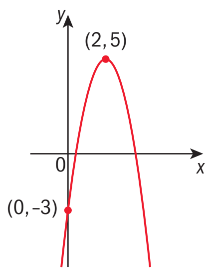
$f(x) = 3(x-1)^2 + 4$



d $f(x) = -2x^2 + 8x - 3 = -2(x^2 - 4x) - 3$

$= -2(x^2 - 4x + 4) - 3 + 8$

$f(x) = -2(x-2)^2 + 5$



Exercise 2I

1 **a** x-intercepts $(-3, 0)$ and $(7, 0)$

$f(x) = (x+3)(x-7) = x^2 - 4x - 21$

y-intercept $(0, -21)$

b x-intercepts $(4, 0)$ and $(5, 0)$

$f(x) = 2(x-4)(x-5) = 2x^2 - 18x + 40$

y-intercept $(0, 40)$

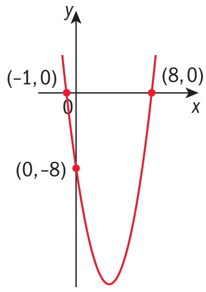
c x-intercepts $(-2, 0)$ and $(-1, 0)$

$f(x) = -3(x+2)(x+1) = -3x^2 - 9x - 6$

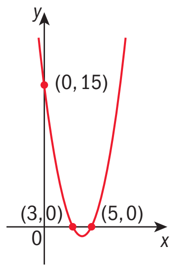
y-intercept $(0, -6)$

d x-intercepts $(-6, 0)$ and $(2, 0)$
$f(x) = 5(x + 6)(x - 2) = 5x^2 + 20x - 60$
y-intercept $(0, -60)$

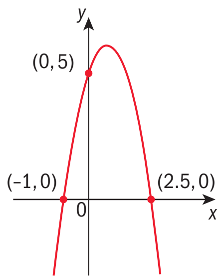
2 a $y = (x - 8)(x + 1)$



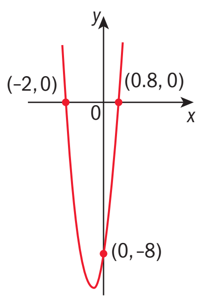
b $y = (x - 3)(x - 5)$



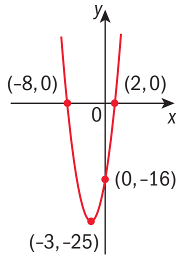
c $y = -2x^2 + 5x - 2x + 5$
$= x(-2x + 5) + 1(-2x + 5)$
$= (x + 1)(-2x + 5)$
$= -2(x + 1)(x - 2.5)$



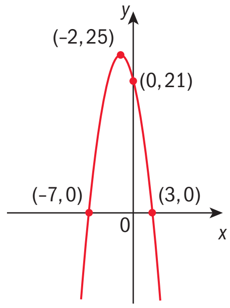
d $y = 5x^2 + 10x - 4x - 8$
$= 5x(x + 2) - 4(x + 2)$
$= (5x - 4)(x + 2)$
$= 5(x - 0.8)(x + 2)$



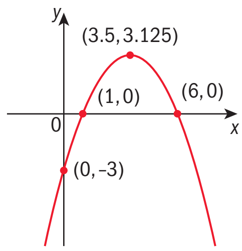
3 a $y = (x + 8)(x - 2)$
$y = x^2 + 6x - 16 = (x^2 + 6x + 9) - 16 - 9$
$= (x + 3)^2 - 25$



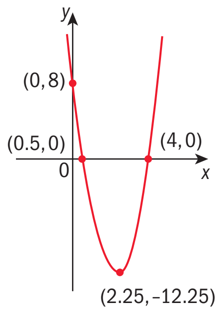
b $y = -(x^2 + 4x - 21) = -(x + 7)(x - 3)$
$y = -x^2 - 4x + 21 = -(x^2 + 4x + 4) + 21 + 4$
$= -(x + 2)^2 + 25$



c $y = -\frac{1}{2}(x^2 - 7x + 6)$
$= -0.5(x - 1)(x - 6)$
$y = -0.5x^2 + 3.5x - 3$
$= -0.5(x^2 - 7x + 12.25) - 3 + 6.125$
$= -0.5(x - 3.5)^2 + 3.125$



d $y = (4x - 2)(x - 4) = 4(x - 0.5)(x - 4)$
$y = 4(x^2 - 4.5x + 5.0625) + 8 - 20.25$
$= 4(x - 2.25)^2 - 12.25$



4 **a** **i** and **a** **ii**

$f(x) = 2x^2 - 12x = 2x(x - 6)$

function crosses x-axis at 0 and 6

b $x = \frac{12}{2(2)} = \frac{12}{4} = 3$

c We know that the axis of symmetry passes through the vertex. So, to find the y-co-ordinates of the vertex, evaluate $f(x)$ at $x = 3$.

$f(3) = 2(3)^2 - 12(3)$

$= 18 - 36 = -18$

vertex at $(3, -18)$

the vertex could also be found by writing the equation in turning-point form.

5 **a** $(f \circ g)(x) = (x - 2)^2 + 3$

b $(2, 3)$

c $h(x) = ((x - 2) - 5)^2 + 3 - 2 = (x - 7)^2 + 1$

$= x^2 - 14x + 50$

d $(0, 50)$

Exercise 2J

1 $y = a(x - 2)^2 + 1$

$5 = a(0 - 2)^2 + 1 \rightarrow 4a = 4 \rightarrow a = 1$

$y = 1(x - 2)^2 + 1 = x^2 - 4x + 5$

2 $y = a(x + 2)(x - 6)$

$-12 = a(0 + 2)(0 - 6) \rightarrow -12a = -12 \rightarrow a = 1$

$y = 1(x + 2)(x - 6) = x^2 - 4x - 12$

3 $y = a(x + 1)^2 + 8$

$5 = a(0 + 1)^2 + 8 \rightarrow a = -3$

$y = -3(x + 1)^2 + 8 = -3x^2 - 6x + 5$

4 $y = a(x + 1)(x - 6)$

$-5 = a(4 + 1)(4 - 6) \rightarrow -10a = -5 \rightarrow a = \frac{1}{2}$

$y = \frac{1}{2}(x + 1)(x - 6) = \frac{1}{2}x^2 - \frac{5}{2}x - 3$

5 at $(-4, 8)$, $a(-4)^2 + b(-4) + c = 16a - 4b + c = 8$

at $(0, 4)$, $a(0)^2 + b(0) + c = 4$

at $(1, 13)$, $a(1)^2 + b(1) + c = a + b + c = 13$

using GDC, $a = 2$, $b = 7$, $c = 4$

$y = 2x^2 + 7x + 4$

6 at $(5, 30)$, $a(5)^2 + b(5) + c = 25a + 5b + c = 30$

at $(15, 30)$, $a(15)^2 + b(15) + c = 225a + 15b + c = 30$

at $(20, 0)$, $a(20)^2 + b(20) + c = 400a + 20b + c = 0$

using GDC, $a = -0.4$, $b = 8$, $c = 0$

$y = -0.4x^2 + 8x$

7 $y = a(x - 2)^2 + 25$

$0 = a(7 - 2)^2 + 25 \rightarrow 25a = -25 \rightarrow a = -1$

$y = -(x - 2)^2 + 25 = -x^2 + 4x + 21$

Alternatively,

$y = a(x + 3)(x - 7)$

$25 = a(2 + 3)(2 - 7) \rightarrow -25a = 25$

$a = -1$

$y = -1(x + 3)(x - 7) = -x^2 + 4x + 21$

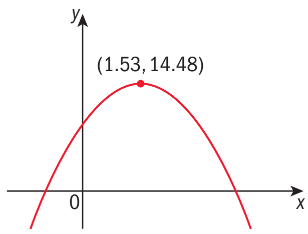
8 $y = a(x - 0.5)^2$

$3 = a(1 - 0.5)^2 \rightarrow 0.25a = 3 \rightarrow a = 12$

$y = 12(x - 0.5)^2 = 12x^2 - 12x + 3$

Exercise 2K

1 **a** Graph $y = 15x - 4.9x^2 + 3$, and find the maximum point (vertex).



maximum height is approximately 14.5 meters.

b $12 = 15t - 4.9t^2 + 3$

$4.9t^2 - 15t + 9 = 0$

$t \approx 0.82, 2.24$

approximately 1.42 seconds.

2 $252 = 32x - x^2$

$x^2 - 32x + 252 = 0$

$(x - 14)(x - 18) = 0$

14 cm, 18 cm

3 **a** The perimeter of square of sides x cm is $4x$ cm.

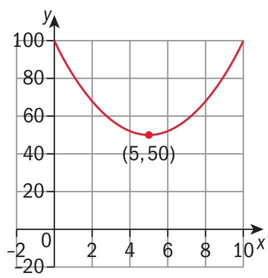
The perimeter of the other square is $40 - 4x$ cm,

So it has sides of length $\frac{40 - 4x}{4} = 10 - x$

b $A = x^2 + (10 - x)^2 = x^2 + 100 - 20x + x^2$

$= 2x^2 - 20x + 100$

c graph $y = 2x^2 - 20x + 100$, and find minimum point (vertex)



minimum combined area is 50 cm^2

4 Let x be the width of the frame. We will subtract the area of the smaller rectangle from the area of the larger one.

$(50 + 2x)(70 + 2x) - (50)(70) = (50)(70)$

$3500 + 240x + 4x^2 - 3500 = 3500$

$4x^2 + 240x - 3500 = 0$

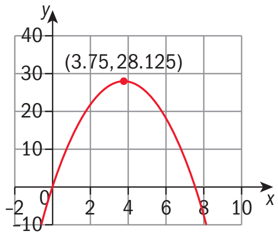
approximately 12.1 cm

5 $l = 3w - 5$
$lw = (3w - 5)w = 3w^2 - 5w = 782$
$3w^2 - 5w - 782 = 0$
$w = 17$
17 m, 46 m

6 $x^2 + (x + 2)^2 + (x + 4)^2 = 251$
$x^2 + x^2 + 4x + 4 + x^2 + 8x + 16 = 251$
$3x^2 + 12x - 231 = 0$
$x = 7$
7, 9, 11

7 Let x = AB.
Then PB = AB − PB
$= x - 1$
Since $\frac{AB}{AD} = \frac{BC}{PB}$,
$\frac{x}{1} = \frac{1}{x-1}$
$x^2 - x - 1 = 0$
$x = \frac{1+\sqrt{5}}{2}$

8 Let x represent the width of the deck, and y represent the area of the deck.
$y = x(15 - 2x) = 15x - 2x^2$



maximum area is 28.125 m^2

9 Let x represent the average speed of the bus.
$\frac{360}{x} + \frac{140}{x+10} = 8$
$360 + \frac{140x}{x+10} = 8x$
$360x + 3600 + 140x = 8x^2 + 80x$
$8x^2 - 420x - 3600 = 0$
$x = 60$
bus 60 km/h, train 70 km/h

10 Let x represent the time it takes John to clean the house.
$\frac{1}{x} + \frac{1}{x-2} = \frac{1}{2.4}$
$1 + \frac{x}{x-2} = \frac{x}{2.4}$
$x - 2 + x = \frac{x^2 - 2x}{2.4}$
$4.8x - 4.8 = x^2 - 2x$
$x^2 - 6.8x + 4.8 = 0$
$x = 6$
It takes John 6 hours to clean the house.

Review exercise

1 **a** $x + 2 = \pm\sqrt{16} = \pm 4$
$x = -2 \pm 4$
$x = -6, 2$

b $(x - 8)^2 = 0$
$x = 8$

c $3x^2 + 7x - 3x - 7 = 0$
$x(3x + 7) - 1(3x + 7) = 0$
$(x - 1)(3x + 7) = 0$
$x = -\frac{7}{3}, 1$

d $(x - 3)(x - 4) = 0$
$x = 3, 4$

e $x^2 + 2x = 12$
$x^2 + 2x + 1 = 12 + 1$
$(x + 1)^2 = 13$
$x + 1 = \pm\sqrt{13}$
$x = -1 \pm \sqrt{13}$

f $x = \frac{7 \pm \sqrt{(7)^2 - 4(3)(3)}}{2(3)} = \frac{7 \pm \sqrt{13}}{6}$

2 **a** -4

b $f(x) = (x + 4)(x - 1)$
$-4, 1$

c $x = \frac{-3}{2(1)} = \frac{-3}{2}$

d $\frac{-3}{2}$

3 **a** $-5, 1$

b $10 = a(0 + 5)(0 - 1)$
$10 = -5a$
$a = -2$

4 **a** $(-3, -6)$

b $2 = a(1 + 3)^2 - 6$
$8 = 16a$
$a = \frac{1}{2}$

c $f(x) = \frac{1}{2}(x+3)^2 - 6$
$f(3) = \frac{1}{2}(3+3)^2 - 6 = 18 - 6$
$f(3) = 12$

5 $(2k)^2 - 4(1)(3) = 0$
$4k^2 = 12$
$k^2 = 3$
$k = \pm\sqrt{3}$

6 **a** $f(x) = 2(x^2 + 6x + 9) + 5 - 18$
$f(x) = 2(x + 3)^2 - 13$

b vertex of f is $(-3, -13)$.
shift 4 to the right, and 8 up, vertex of g is $(1, -5)$

7 $y = a(x+4)(x-6)$
$-12 = a(2+4)(2-6)$
$-12 = -24a$
$a = \frac{1}{2}$
$y = \frac{1}{2}(x+4)(x-6) = \frac{1}{2}x^2 - x - 12$

Review exercise

1 **a** $x = -0.907, 2.57$

b $2x^2 + 8x - 3 = 0$
$x = -4.35, 0.345$

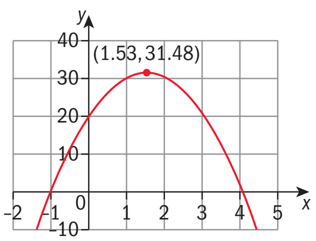
c $x = (2x-1)(x+3)$
$x = 2x^2 + 5x - 3$
$2x^2 + 4x - 3 = 0$
$x = -2.58, 0.581$

d $1 + \frac{x}{x+2} = 5x$
$x + 2 + x = 5x^2 + 10x$
$5x^2 + 8x - 2 = 0$
$x = -1.82, 0.220$

for question 1, the equations may be solved using polynomial equation solver on the GDC, or by using completing the square or the quadratic formula.

2 **a** Initially, $t = 0$, so
$h(0) = 15(0) + 20 - 4.9(0)^2 = 20$
20 m

b graph $y = 15x + 20 - 4.9x^2$



31.5 m

c $20 = 15t + 20 - 4.9t^2$
$4.9t^2 - 15t = 0$
$t = 0, t \approx 3.06$
3.06 s

d The stone hits the water when $h = 0$
$0 = 15t + 20 - 4.9t^2$
4.07 s

3 $l = 3w + 5$
$lw = (3w+5)w = 3w^2 + 5w = 1428$
$3w^2 + 5w - 1428 = 0$
$w = 21, l = 68$

4 at $(-10, 12)$, $a(-10)^2 + b(-10) + c$
$= 100a - 10b + c = 12$
at $(-5, -3)$, $a(-5)^2 + b(-5) + c$
$= 25a - 5b + c = -3$
at $(5, 27)$, $a(5)^2 + b(5) + c$
$= 25a + 5b + c = 27$
using GDC, $a = 0.4$, $b = 3$, $c = 2$

5 Let x represent his average driving speed.
Since time $= \frac{\text{distance}}{\text{speed}}$, and the difference in the two times is half an hour,

$$\frac{120}{x} = \frac{120}{x+20} + 0.5$$

$$120 = \frac{120x}{x+20} + 0.5x$$

$120x + 2400 = 120x + 0.5x^2 + 10x$
$0.5x^2 + 10x - 2400 = 0$
$x = 60$
60 kmh^{-1}

3 Probability

Answers

Skills check

1 **a** $1-\frac{3}{7}=\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$

b $\frac{2}{5}+\frac{5}{7}=\frac{14+25}{35}=\frac{39}{35}=1\frac{4}{35}$

c $\frac{1}{5}\times\frac{2}{3}=\frac{2}{15}$

d $1-\left(\frac{1}{3}\times\frac{5}{9}\right)=1-\frac{5}{27}=\frac{27-5}{27}=\frac{22}{27}$

e $\dfrac{\frac{3}{20}}{\frac{7}{20}}=\frac{3}{20}\times\frac{20}{7}=\frac{3}{7}$

2 **a** $1-0.375=0.625$

b $0.65+0.05=0.7$

c $0.7\times 0.6=0.42$

d $0.25\times 0.64=0.16$

e 50% of 30 = $0.5\times 30=15$

f 22% of 0.22 = $0.22\times 0.22=0.0484$

g 12% of 10% of 0.8 = $0.12\times 0.1\times 0.8=0.0096$

Exercise 3A

1 **a** $P(2, 4, 6, 8)=\frac{4}{8}=\frac{1}{2}$

b $P(3, 6)=\frac{2}{8}=\frac{1}{4}$

c $P(4, 8)=\frac{2}{8}=\frac{1}{4}$

d $P(1, 2, 3, 5, 6, 7)=\frac{6}{8}=\frac{3}{4}$ or

$1-P(4, 8)=1-\frac{1}{4}=\frac{3}{4}$

e $P(1, 2, 3)=\frac{3}{8}$

2 P(defective car) $=\frac{\text{number defective}}{\text{number of cars}}=\frac{30}{150}=\frac{1}{5}$

3 **a** **i** 0.21

ii $0.19+0.14=0.33$

b Proportion of 15 year old students = 0.21

Therefore $0.21\times 1200=252$ students who are 15.

4 **a** $\frac{27}{100}=0.27$

b No – the frequencies for different numbers are very different

c $\frac{15}{100}\times 3000=450$

5 **a** $\frac{\text{number of c's}}{\text{number of letters}}=\frac{2}{11}$

b $\frac{\text{number of p's}}{\text{number of letters}}=\frac{0}{11}=0$

c $\frac{\text{number of vowels}}{\text{number of letters}}=\frac{5}{11}$

6 P(red) + P(yellow) + P(green) + P(blue) = 1

Let P(yellow) = x so P(green) = $2x$

$0.4+x+2x+0.3=1$

$3x=0.3$

$x=0.1$

Therefore P(green) = 0.2

7 **a** $\frac{\text{number of even numbers}}{\text{number of possible outcomes}}=\frac{20}{40}=\frac{1}{2}$

b {1, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 31}

$\frac{\text{number that contain digit 1}}{\text{number of possible outcomes}}=\frac{13}{40}$

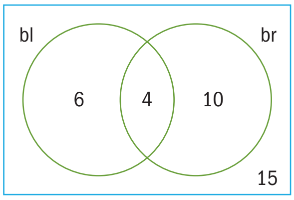
Exercise 3B

1 n(blond and brown) = 4

n(blond and not brown) = 10 – 4 = 6

n(brown and not blond) = 14 – 4 = 10

n(neither blond or brown) = 35 – (6 + 4 + 10) = 15



P(blond hair or blue eyes) $=\frac{6+4+10}{35}=\frac{20}{35}=\frac{4}{7}$

2 n(French and Malay) = x

n(F and not M) = $15-x$

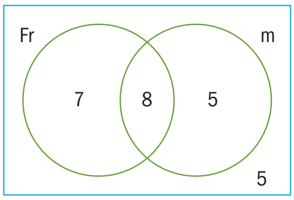
n(M and not F) = $13-x$

n(neither F or M) = 5

Therefore $x+(15-x)+(13-x)+5=25$

$33-x=25$

$x=8$



P(F and M) $=\frac{8}{25}$

3 n(Aerobics and Gymnastics) = x

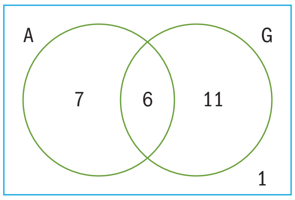
n(A and not G) = $13-x$

n(G and not A) = $17-x$

n(neither A or G) = 1

Therefore $x+(13-x)+(17-x)+1=25$

$31 - x = 25$
$x = 6$



a P(A and G) $= \frac{6}{25}$

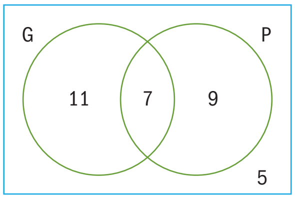
b P(G and not A) $= \frac{11}{25}$

4 n(Golf and Piano) = 7

n(G and not P) = 18 – 7 = 11

n(P and not G) = 16 – 7 = 9

n(neither G or P) = 32 – (7 + 11 + 9) = 5



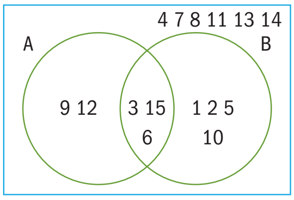
a P(G and not P) $= \frac{11}{32}$

b P(P and not G) $= \frac{9}{32}$

5 **a** A = {integers that are multiples of 3}
= { 3, 6, 9, 12, 15}

B = {integers that are factors of 30}
= {1, 2, 3, 5, 6, 10, 15}

b



c **i** P(both a multiple of 3 and a factor of 30) $= \frac{3}{15} = \frac{1}{5}$

ii P(Neither a multiple of 3 or a factor of 30) $= \frac{6}{15} = \frac{2}{5}$

6 n(A & B, not C) = 5% − 2% = 3%

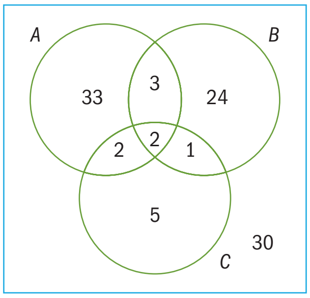
n(A & C, not B) = 4% − 2% = 2%

n(B & C, not A) = 3% − 2% = 1%

n(A, not B or C) = 40% − (2% + 3% + 2%) = 33%

n(B, not A or C) = 30% − (2% + 3% + 1%) = 24%

n(C, not A or B) = 10% − (2% + 2% + 1%) = 5%



a P(only A) = 0.33

b P(only B) = 0.24

c P(none) = 0.3

Exercise 3C

1 **a** $\frac{\text{number that are divisible by 5}}{\text{number of possible outcomes}} = \frac{\text{frequencies of } \{5, 10\}}{\text{number of possible outcomes}}$

$= \frac{34+68}{500} = \frac{102}{500} = \frac{51}{250}$

b $\frac{\text{number that are even}}{\text{number of possible outcomes}}$

$= \frac{\text{frequencies of } \{2, 4, 6, 8, 10, 12\}}{\text{number of possible outcomes}}$

$= \frac{6+21+65+63+68+42}{500} = \frac{265}{500} = \frac{53}{100}$

c $\frac{\text{number that are divisible by 5 or even}}{\text{number of possible outcomes}}$

$= \frac{\text{frequencies of } \{2,4,5,6,8,10,12\}}{\text{number of possible outcomes}}$

$= \frac{6+21+65+63+68+42+34}{500} = \frac{299}{500}$

or P(sum divisible by 5 $\cup$ sum even)

= P(sum divisible by 5) + P(sum even)

− P(sum divisible by 5 $\cap$ sum even)

$= \frac{102}{500} + \frac{265}{500} - \frac{68}{500} = \frac{299}{500}$

2 **a** P(prime) $= \frac{4}{10} = \frac{2}{5}$ [primes are 2, 3, 5, 7]

b P(prime or multiple of 3) $= \frac{4}{10} + \frac{3}{10} - \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$

c P(multiple of 3 or 4) $= \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$

3 P(camera owner or female)

= P(camera owner) + P(female)

− P(female camera owner)

$= \frac{40}{80} + \frac{50}{80} - \frac{22}{80} = \frac{68}{80} = \frac{17}{20}$

4 **a** 8 different letters in MATHEMATICS{M, A, T, H, E, I, C, S}. $\frac{8}{26} = \frac{4}{13}$

b 9 different letters in TRIGONOMETRY {T, R, I, G, O, N, M, E, Y} $\frac{9}{26}$

c {M, T, E, I} $\frac{4}{26} = \frac{2}{13}$

d {M, A, T, H, E, I, C, S, R, G, O, N, Y} $\frac{13}{26} = \frac{1}{2}$

5 **a** P(work of fiction, non-fiction, or both) = 0.40 + 0.30 − 0.20 = 0.5

b P(no book) = 1 − 0.5 = 0.5

6 Let P(local and national) = x

P(national and not local) = $\frac{1}{4} - x$

P(local and not national) = $\frac{3}{5} - x$

$$\frac{2}{3} = \left(\frac{1}{4} - x\right) + \left(\frac{3}{5} - x\right) + x$$

$$x = \frac{11}{60}$$

7 **a** $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$$= \frac{1}{4} + \frac{1}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{4}$$

b $P(X) \cup P(Y)' = 1 - P(X \cup Y) = 1 - \frac{1}{4} = \frac{3}{4}$

8 **a** $P(A \cup B) = P(A) + P(B) - \mathrm{P}(A \cup B)$

$$= 0.2 + 0.5 - 0.1 = 0.6$$

b $P(A \cup B)' = 1 - P(A \cup B) = 1 - 0.6 = 0.4$

c $P(A' \cup B) = 1 - P(A \cap B')$

$$= 1 - [P(A) - P(A \cap B)]$$

$$= 1 - [0.2 - 0.1] = 0.9$$

Exercise 3D

1 **a** A and B = N **b** A and C = Y

c A and D = N **d** A and E = Y

e B and E = N **f** C and D = N

g B and C = N

2 Let $P(N \cap M) = x$. If $P(N \cap M) = 0$ then N & M are mutually exclusive.

Now $P(N \cup M) = P(N) + P(M) - P(N \cup M)$, so

$$\frac{3}{10} = \frac{1}{5} + \frac{1}{10} - x$$

$$x = 0$$

Therefore yes

3 $\frac{30}{89} + \frac{27}{89} = \frac{57}{89}$

4 **a** $\frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$

b $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{20+15+12}{60} = \frac{47}{60}$

c $1 - \frac{47}{60} = \frac{13}{60}$

Exercise 3E

1 {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

a {HHH, HHT, HTH, THH} $\frac{1}{2}$

b {HHH, HHT, THH} $\frac{3}{8}$

c {HTH,THT} $\frac{1}{4}$

2

		BLUE			
		1	2	3	4
RED	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

a $\frac{6}{16} = \frac{3}{8}$

		BLUE			
		1	2	3	4
RED	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

b $\frac{6}{16} = \frac{3}{8}$

		BLUE			
		1	2	3	4
RED	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

c $\frac{4}{16} = \frac{1}{4}$

		BLUE			
		1	2	3	4
RED	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

d $\frac{9}{16}$

		BLUE			
		1	2	3	4
RED	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

3

		Box 1		
		1	2	3
Box 2	2	(2, 1)	(2, 2)	(2, 3)
	3	(3, 1)	(3, 2)	(3, 3)
	4	(4, 1)	(4, 2)	(4, 3)
	5	(5, 1)	(5, 2)	(5, 3)

a $\frac{2}{12} = \frac{1}{6}$

		Box 1		
		1	2	3
Box 2	2	(2, 1)	(2, 2)	(2, 3)
	3	(3, 1)	(3, 2)	(3, 3)
	4	(4, 1)	(4, 2)	(4, 3)
	5	(5, 1)	(5, 2)	(5, 3)

b $\frac{3}{12} = \frac{1}{4}$

		Box 1		
		1	2	3
Box 2	2	(2, 1)	(2, 2)	**(2, 3)**
	3	**(3, 1)**	**(3, 2)**	(3, 3)
	4	(4, 1)	(4, 2)	(4, 3)
	5	(5, 1)	(5, 2)	(5, 3)

c $\frac{9}{12} = \frac{3}{4}$

		Box 1		
		1	2	3
Box 2	2	**(2, 1)**	**(2, 2)**	**(2, 3)**
	3	**(3, 1)**	**(3, 2)**	**(3, 3)**
	4	**(4, 1)**	**(4, 2)**	(4, 3)
	5	**(5, 1)**	(5, 2)	(5, 3)

d $\frac{5}{12}$

		Box 1		
		1	2	3
Box 2	2	(2, 1)	(2, 2)	(2, 3)
	3	(3, 1)	(3, 2)	**(3, 3)**
	4	(4, 1)	**(4, 2)**	**(4, 3)**
	5	(5, 1)	**(5, 2)**	**(5, 3)**

e $\frac{8}{12} = \frac{2}{3}$

		Box 1		
		1	2	3
Box 2	2	**(2, 1)**	**(2, 2)**	**(2, 3)**
	3	(3, 1)	**(3, 2)**	(3, 3)
	4	**(4, 1)**	**(4, 2)**	**(4, 3)**
	5	(5, 1)	**(5, 2)**	(5, 3)

4

		First draw					
		0	1	2	3	4	5
Second draw	0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)
	1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
	2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
	3	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
	4	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
	5	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

a $\frac{6}{36} = \frac{1}{6}$

		First draw					
		0	1	2	3	4	5
Second draw	0	**(0, 0)**	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)
	1	(1, 0)	**(1, 1)**	(1, 2)	(1, 3)	(1, 4)	(1, 5)
	2	(2, 0)	(2, 1)	**(2, 2)**	(2, 3)	(2, 4)	(2, 5)
	3	(3, 0)	(3, 1)	(3, 2)	**(3, 3)**	(3, 4)	(3, 5)
	4	(4, 0)	(4, 1)	(4, 2)	(4, 3)	**(4, 4)**	(4, 5)
	5	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	**(5, 5)**

b $\frac{20}{36} = \frac{5}{9}$

		First draw					
		0	1	2	3	4	5
Second draw	0	(0, 0)	(0, 1)	**(0, 2)**	**(0, 3)**	(0, 4)	**(0, 5)**
	1	(1, 0)	(1, 1)	**(1, 2)**	**(1, 3)**	(1, 4)	**(1, 5)**
	2	**(2, 0)**	**(2, 1)**	(2, 2)	**(2, 3)**	(2, 4)	**(2, 5)**
	3	**(3, 0)**	**(3, 1)**	**(3, 2)**	(3, 3)	(3, 4)	**(3, 5)**
	4	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	**(4, 5)**
	5	**(5, 0)**	**(5, 1)**	**(5, 2)**	**(5, 3)**	**(5, 4)**	(5, 5)

c $\frac{26}{36} = \frac{13}{18}$

		First draw					
		0	1	2	3	4	5
Second draw	0	**(0, 0)**	**(0, 1)**	**(0, 2)**	**(0, 3)**	**(0, 4)**	**(0, 5)**
	1	**(1, 0)**	**(1, 1)**	**(1, 2)**	**(1, 3)**	**(1, 4)**	**(1, 5)**
	2	**(2, 0)**	**(2, 1)**	**(2, 2)**	**(2, 3)**	**(2, 4)**	(2, 5)
	3	**(3, 0)**	**(3, 1)**	**(3, 2)**	**(3, 3)**	(3, 4)	(3, 5)
	4	**(4, 0)**	**(4, 1)**	**(4, 2)**	(4, 3)	(4, 4)	(4, 5)
	5	**(5, 0)**	**(5, 1)**	(5, 2)	(5, 3)	(5, 4)	(5, 5)

d $\frac{12}{36} = \frac{1}{3}$

		First draw					
		0	1	2	3	4	5
Second draw	0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)
	1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
	2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	**(2, 4)**	**(2, 5)**
	3	(3, 0)	(3, 1)	(3, 2)	(2, 3)	**(3, 4)**	**(3, 5)**
	4	(4, 0)	(4, 1)	**(4, 2)**	**(4, 3)**	**(4, 4)**	**(4, 5)**
	5	(5, 0)	(5, 1)	**(5, 2)**	**(5, 3)**	**(5, 4)**	**(5, 5)**

e $\frac{27}{36} = \frac{3}{4}$

		First draw					
		0	1	2	3	4	5
Second draw	0	**(0, 0)**	**(0, 1)**	**(0, 2)**	**(0, 3)**	**(0, 4)**	**(0, 5)**
	1	**(1, 0)**	(1, 1)	**(1, 2)**	(1, 3)	**(1, 4)**	(1, 5)
	2	**(2, 0)**	**(2, 1)**	**(2, 2)**	**(2, 3)**	**(2, 4)**	**(2, 5)**
	3	**(3, 0)**	(3, 1)	**(3, 2)**	(3, 3)	**(3, 4)**	(3, 5)
	4	**(4, 0)**	**(4, 1)**	**(4, 2)**	**(4, 3)**	**(4, 4)**	**(4, 5)**
	5	**(5, 0)**	(5, 1)	**(5, 2)**	(5, 3)	**(5, 4)**	(5, 5)

5 When rolling the dice twice, there are 36 possible outcomes

a {(1, 3), (2,4), (3, 1), (4, 2), (5, 5), (5, 6), (6, 5), (6, 6)}; $\frac{8}{36} = \frac{2}{9}$

b {(1, 1), (2, 2), (3, 3), (4, 4)}; $\frac{4}{36} = \frac{1}{9}$

c {(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3)}; $\frac{8}{36} = \frac{2}{9}$

Exercise 3F

1 $\frac{1}{5}\times\frac{1}{5}=\frac{1}{25}$

2 $P(K)\times P(10)=\frac{4}{52}\times\frac{4}{52}=\frac{1}{169}$

3 $\left(\frac{4}{5}\right)^3=\frac{64}{125}$

4 $P(C)\times P(H)=0.75\times0.85=0.6375=0.638$

5 **a** $P(A\cup B)=P(A)+P(B)-P(A\cap B)$
Let $P(B)=x$.
$0.4=0.2+x-0$
$x=0.2$
$P(B)=0.2$

$P(B\cup C)=P(B)+P(C)-P(B\cap C)$
Let $P(B\cap C)=y$.
$0.34=0.2+0.3-y$
$y=0.16$
$P(B\cap C)=0.16$

b $P(B)\times P(C)=0.2\times0.3=0.06$
$P(B\cap C)=0.16$
$P(B\cap C)\neq P(B)\times P(C)$
Not independent

6 $P(H)\times P(\bar{6})=\frac{1}{2}\times\frac{5}{6}=\frac{5}{12}$

7 $\left(\frac{1}{9}\right)^5=\frac{1}{59049}$

8 $P(H)=\frac{1}{4}$, therefore for 4 hearts $\left(\frac{1}{4}\right)^4=\frac{1}{256}$

9 **a** $P(E)=1-P(E')=1-0.6=0.4$

b $P(E)\times P(F)=0.4\times0.6=0.24=P(E\cap F)$

c $P(E\cap F)\neq 0$

d $P(E\cup F')=P(E)+P(F')-P(E\cap F')$
We know that since E & F are independent,
$P(E\cap F')=P(E)\times P(F')=0.4\times0.4$
$P(E\cup F')=0.4+0.4-(0.4\times0.4)=0.64$

10 $P(R_1 \text{ and } B_2 \text{ and } R_3)=\frac{4}{12}\times\frac{8}{12}\times\frac{4}{12}=\frac{2}{27}$

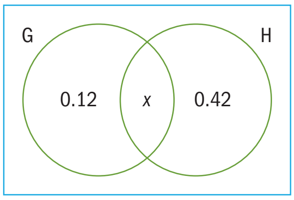
11 $\{2, 2, 2\}$; $\left(\frac{1}{3}\right)^3=\frac{1}{27}$

12 **a** $P(A\cap B)=P(A)\times P(B)=0.9\times0.3=0.27$
since A & B are independent

b $P(A\cap B')=0.9\times0.7=0.63$
(since $P(B')=1-P(B)=0.7$)

c $P(A\cup B)'=1-P(A\cup B)$
$=1-[P(A)+P(B)-P(A\cap B)]$
$=1-(0.9+0.3-0.27)$
$=0.07$

13



G & H are independent, so $P(G\cap H)=P(G)\times P(H)$
Now $P(G)=0.12+x$, and $P(H)=0.42+x$, so
$(0.12+x)(0.42+x)=x$
$x^2-0.46x+0.0504=0$
$x=0.18, 0.28$

14 **a** $P(4\text{ sixes})=\left(\frac{1}{6}\right)^4=\frac{1}{1296}$

b $P(4\text{ same})=6\left(\frac{1}{6}\right)^4=\frac{6}{1296}=\frac{1}{216}$

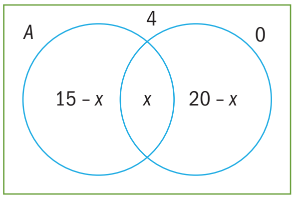
15 rolling a 'six' on four throws of one dice:
P(rolling a 'six' on four throws of one dice) $=1-\left(\frac{5}{6}\right)^4=1-\frac{625}{1296}=\frac{671}{1296}=0.518$
P(rolling a 'double six' on 24 throws with two dice) $=1-\left(\frac{35}{36}\right)^{24}=1-0.5085\ldots=0.491$

16 **a** P(not a 5) = 0.9. we require $(0.9)^3=0.729$

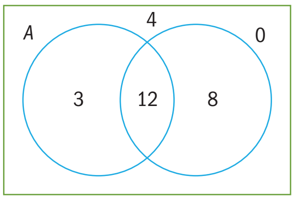
b $1-$ P(none is a 5) $=1-0.729=0.271$

Exercise 3G

1 Let $n(A\cap D)=x$



$15-x+x+20-x+4=27$
$39-x=27$
$x=12$



a P(Drama not Art) $=\frac{8}{27}$

b P(Takes at least one of the two subjects)
$=1-$ P(takes none) $=1-\frac{4}{27}=\frac{23}{27}$

c P(Takes both subjects, given that he takes Art)
$=\dfrac{\frac{12}{27}}{\frac{15}{27}}=\frac{12}{15}=\frac{4}{5}$

2 **a** $P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.35 = 0.65$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.65 = 0.25 + 0.6 - P(A \cap B)$

$P(A \cap B) = 0.85 - 0.65 = 0.2$

b $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{0.2}{0.6}$

$= \frac{1}{3}$

c $P(B'|A') = \frac{P(B' \cap A')}{P(A')}$

$= \frac{0.35}{0.75}$

$= \frac{7}{15}$

3 $P(R|S) = \frac{P(R \cap S)}{P(S)}$

$= \frac{0.39}{0.48}$

$= 0.8125 = \frac{13}{16}$

4 **a** $P(E|M') = \frac{P(E \cap M')}{P(M')}$

$= \frac{0.25}{0.75}$

$= \frac{1}{3}$

b $P(<15|>5) = \frac{P(<15 \cap >5)}{P(>5)}$

$= \frac{\frac{2}{8}}{\frac{5}{8}}$

$= \frac{2}{5}$

c $P(<5|<15) = \frac{P(<5 \cap <15)}{P(<15)}$

$= \frac{\frac{3}{8}}{\frac{5}{8}}$

$= \frac{3}{5}$

d $P(\text{between } 10 \text{ and } 20|\text{between } 5 \text{ and } 25)$

$= \frac{P(\text{between } 10 \text{ and } 20 \text{ and between } 5 \text{ and } 25)}{P(\text{between } 5 \text{ and } 25)}$

$= \frac{\frac{2}{8}}{\frac{4}{8}}$

$= \frac{1}{2}$

5 $P(L|D) = \frac{P(D \cap L)}{P(D)} = \frac{0.61}{0.95} = \frac{61}{95}$

6 $P(S|T) = \frac{P(T \cap S)}{P(T)} = \frac{0.1}{0.6} = \frac{1}{6}$

7 **a** P(U and V) = 0 by definition

b P(U | V) = 0 by definition

c P(U or V) = P(U) + P(V) = 0.26 + 0.37 = 0.63

8 $\frac{P(\text{Pass both})}{P(\text{Pass first})} = \frac{0.35}{0.52} = 0.673$. Therefore 67.3%

9 $P(B_1 \text{ and } W_2) = 0.34$; $P(B_1) = 0.47$.

$P(W_2|B_1) = \frac{P(B_1 \text{ and } W_2)}{P(B_1)} = \frac{0.34}{0.47} = \frac{34}{47}$

10 **a** P(male and left handed) = $\frac{5}{50} = \frac{1}{10}$

	Left handed	Right handed	Total
Male	5	32	37
Female	2	11	13
Total	7	43	50

b P(right handed) = $\frac{43}{50}$

	Left handed	Right handed	Total
Male	5	32	37
Female	2	11	13
Total	7	43	50

c P(right handed, given that the player selected is female) = $\frac{11}{13}$

	Left handed	Right handed	Total
Male	5	32	37
Female	2	11	13
Total	7	43	50

11 $P(J|K) = \frac{P(J \cap K)}{P(K)}$

J & K are independent, so $P(J \cap K) = P(J) \times P(K)$

$\therefore P(J|K) = = \frac{P(J) \times \cancel{P(K)}}{\cancel{P(K)}} = P(J)$

so $P(J) = P(J|K) = 0.3$

12 Let T be the event the neighbor has 2 boys and S be the event that the neighbor has a son

The possible options are {BB, BG, GB, GG}

Event S, the neighbor has a son is the set S={BB, BG, GB}

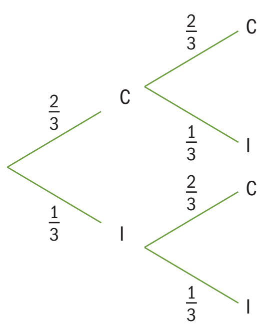
Event T, that the neighbor has two boys is the set T={BB}

We require $P(T|S) = \frac{P(T \cap S)}{P(S)} = \frac{P(\{BB\})}{P(\{BB, BG, GB\})}$

$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Exercise 3H

1 **a**

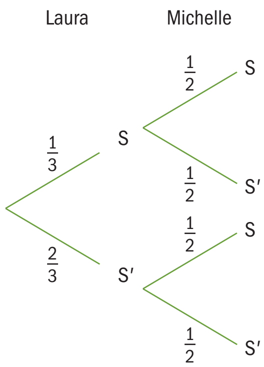


b P(C and I) or P(I and C)

$= \left(\frac{2}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$

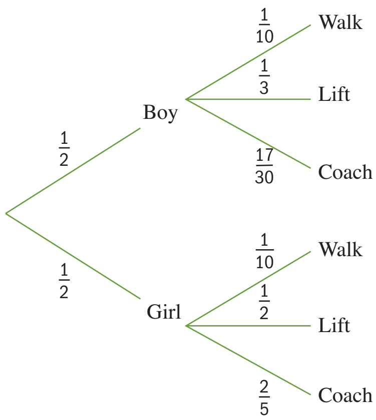
c $1 - \text{P(none correct)} = 1 - \left(\frac{1}{3} \times \frac{1}{3}\right) = 1 - \left(\frac{1}{9}\right) = \frac{8}{9}$

2



P(neither will score in the next game) $= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

3



a $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$

b $\left(\frac{1}{2} \times \frac{2}{5}\right) + \left(\frac{1}{2} \times \frac{17}{30}\right) = \frac{1}{5} + \frac{17}{60} = \frac{29}{60}$

4 P(Head) $= \frac{2}{3}$ We require HHT or HTH or THH. Each has probability $\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$. Therefore

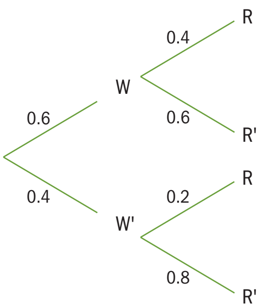
P (HHT or HTH or THH) $= 3 \times \frac{4}{27} = \frac{4}{9}$

5 **a** P(Prime) = 0.4.

P(exactly one Prime) = P(Prime and not prime) + P(not prime and prime) = (0.4 × 0.6) + (0.6 × 0.4) = 0.48

b P(at least one prime) = 1 – P(no primes) = 1 – (0.6 × 0.6) = 1 – 0.36 = 0.64

6 **a**



b P(rainy) = P(W and R) or P(W' and R) = (0.6 × 0.4) + (0.4 × 0.2) = 0.24 + 0.08 = 0.32

c P(two successive days **not** being rainy) = P(not rainy) × P(not rainy)

P(not rainy) = 1 – 0.32 = 0.68

0.68 × 0.68 = 0.4624

Exercise 3I

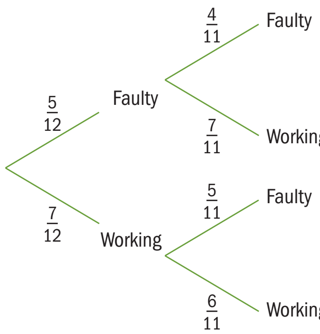
1 **a** P(picture card) $= \frac{12}{52}$

We require $\frac{12}{52} \times \frac{11}{51} \times \frac{10}{50} = \frac{11}{1105}$

b We require $PP\overline{P}$ *or* $P\overline{P}P$ *or* $\overline{P}PP$. Each of these has equal probability of $\frac{12}{52} \times \frac{11}{51} \times \frac{40}{50} = \frac{44}{1105}$

$P(PP\overline{P} \text{ or } P\overline{P}P \text{ or } \overline{P}PP) = 3 \times \frac{44}{1105} = \frac{132}{1105}$

2



a P(two faulty) $= \frac{5}{12} \times \frac{4}{11} = \frac{5}{33}$

b P(exactly one faulty)

$= \left(\frac{5}{12} \times \frac{7}{11}\right) + \left(\frac{7}{12} \times \frac{5}{11}\right) = \frac{35}{132} + \frac{35}{132} = \frac{35}{66}$

c $P(F_2 \mid \text{exactly one faulty pen})$

$$= \frac{P(F_2 \text{ and exactly one faulty pen})}{P(\text{exactly one faulty pen})} = \frac{\frac{7}{12} \times \frac{5}{11}}{\frac{35}{66}} = \frac{1}{2}$$

3 **a** $\frac{3}{9} \times \frac{2}{8} = \frac{1}{12}$

b P(RR or GG or YY) =

$\left(\frac{4}{9} \times \frac{3}{8}\right) + \left(\frac{3}{9} \times \frac{2}{8}\right) + \left(\frac{2}{9} \times \frac{1}{8}\right) = \frac{5}{18}$

c We require P(YY or YG or GG or GY)

$= \left(\frac{2}{9} \times \frac{1}{8}\right) + \left(\frac{2}{9} \times \frac{3}{8}\right) + \left(\frac{3}{9} \times \frac{2}{8}\right) + \left(\frac{3}{9} \times \frac{2}{8}\right) = \frac{5}{18}$

d We require 1 − P(RR or RG or GR or GG)

$= 1 - \left[\left(\frac{4}{9} \times \frac{3}{8}\right) + \left(\frac{4}{9} \times \frac{3}{8}\right) + \left(\frac{3}{9} \times \frac{4}{8}\right) + \left(\frac{3}{9} \times \frac{2}{8}\right)\right]$

$= 1 - \frac{7}{12} = \frac{5}{12}$

4 P(one of each color) = P(RBOP in any order)

P(RBOP) $= \frac{5}{14} \times \frac{4}{13} \times \frac{3}{12} \times \frac{2}{11} = \frac{5}{1001}$

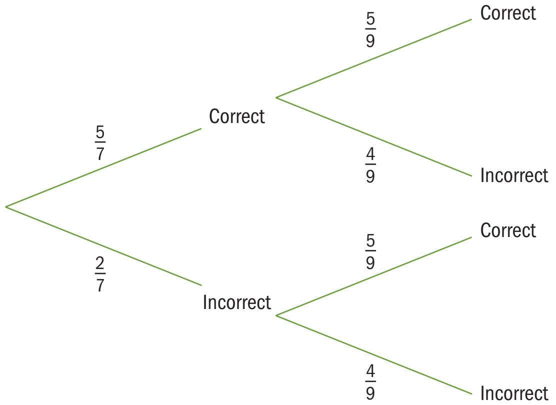
We can arrange RBOP in 24 ways

Therefore required probability $= 24 \times \frac{5}{1001} = \frac{120}{1001}$

5 **a** $\frac{4}{10} = \frac{2}{5}$

b $\left(\frac{4}{10} \times \frac{6}{9}\right) + \left(\frac{6}{10} \times \frac{4}{9}\right) = \frac{8}{15}$

6



a P(at least one of the students answers the question correctly) = 1 – P(both incorrect)

$= 1 - \left(\frac{2}{7} \times \frac{4}{9}\right) = \frac{55}{63}$

b P(Billy correct given that the answer is correct) $= \frac{\frac{5}{7}}{\frac{55}{63}} = \frac{9}{11}$

c P(Natasha correct given that the answer is correct) $= \frac{\frac{5}{9}}{\frac{55}{63}} = \frac{7}{11}$

d P(two correct answers given that there were one) $= \frac{\frac{5}{7} \times \frac{5}{9}}{\frac{55}{63}} = \frac{25}{55} = \frac{5}{11}$

Review exercise

1 There are 90 numbers from 10 to 99 inclusive.

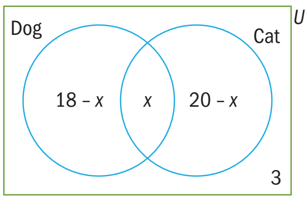
a $\{10, 15, 20, \ldots, 90, 95\}$ or every 5th number is divisible by 5 so $\frac{18}{90} = \frac{1}{5}$

b $\{3, 6, 9, 12, \ldots, 96, 99\}$ or every 3rd number is divisible by 3 so $\frac{1}{3}$

c $\{51, 52, 53, \ldots 98, 99\}$ $\frac{49}{90}$

d $\{16, 25, 36, 49, 64, 81\}$ $\frac{6}{90} = \frac{1}{15}$

2 Let $n(C \cap D) = x$



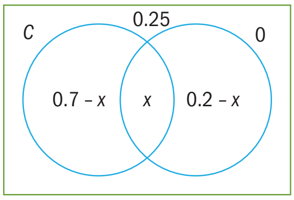
$18 - x + x + 20 - x + 3 = 30$

$41 - x = 30$

$x = 11$

$\text{P(Cat and Dog)} = \frac{11}{30}$

3 Let $\text{P}(C \cap D) = x$



a $0.7 - x + x + 0.2 - x + 0.25 = 1$

$1.15 - x = 1$

$x = 0.15$

$\therefore P(C \cap D') = P(C) - P(C \cap D)$

$= 0.7 - 0.15 = 0.55$

b Not independent since $\text{P}(C \cap D) = 0.15$ and $\text{P}(C) \times \text{P}(D) = 0.7 \times 0.2 = 0.14$

4 **a** We require $\text{P}(A \cap B)$

$\text{P}(A / B) = \frac{\text{P}(A \cap B)}{\text{P}(B)}$

$0.1 = \frac{\text{P}(A \cap B)}{0.2}$

$\text{P}(A \cap B) = 0.1 \times 0.2 = 0.02$

b We require $\text{P}(A \cup B)$

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$\text{P}(A \cup B) = 0.6 + 0.2 - 0.02$

$\text{P}(A \cap B) = 0.78$

c We require

$\text{P}(A \cup B) - \text{P}(A \cap B)$

$= 0.78 - 0.02$

$= 0.76$

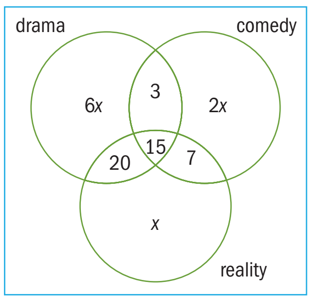
d We require $\text{P}(B \mid A)$

$\text{P}(B|A) = \frac{\text{P}(B \cap A)}{\text{P}(A)}$

$= \frac{0.02}{0.6} = \frac{1}{30} = 0.0333$

5 **a** $6x$

b



c $6x + 3 + 2x + 20 + 15 + 7 + x + 10 = 100$

$9x + 55 = 100$

$9x = 45$

$x = 5$

Review exercise

1 **a** $P(C \cap D) = P(C|D) \times P(D)$
$= 0.6 \times 0.5$
$= 0.3$

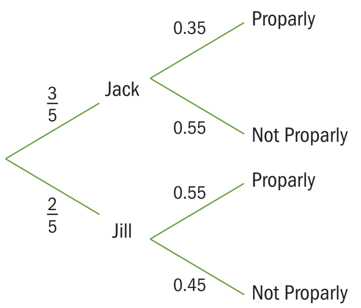
b No since $P(C \text{ and } D) \neq 0$

c No since $P(C) \times P(D) = 0.4 \times 0.5 = 0.2 \neq P(C \text{ and } D)$

d $P(C \cup D) = P(C) + P(D) - P(C \cap D)$
$= 0.4 + 0.5 - 0.3$
$= 0.6$

e $P(D|C) = \frac{P(D \cap C)}{P(C)}$
$= \frac{0.3}{0.4} = 0.75$

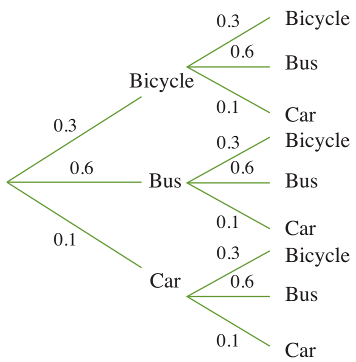
2



a P(Properly) = P(Jack and Properly) + P(Jill and Properly) $= \left(\frac{3}{5} \times 0.35\right) + \left(\frac{2}{5} \times 0.55\right)$
$= 0.21 + 0.22 = 0.43$

b $P(\text{Jill} | \text{Not Properly}) = \frac{P(\text{Jill and not properly})}{P(\text{Not Properly})}$
$= \frac{\frac{2}{5} \times 0.45}{0.57} = 0.316$

3 **a**



b **i** P(Travels by bicycle on Monday and Tuesday) = $0.3 \times 0.3 = 0.09$

ii P(Travels by bicycle on Monday and by bus on Tuesday) = $0.3 \times 0.6 = 0.18$

iii P(Travels by the same method of travel on Monday and Tuesday) = $(0.3 \times 0.3) + (0.6 \times 0.6) + (0.1 \times 0.1) = 0.46$

c P(not by bicycle on 3 days) = $0.7 \times 0.7 \times 0.7$
$= 0.343$

d P(twice by car & once by bus)
= P(car ∩ car ∩ bus) + P(car ∩ bus ∩ car)
+ P(bus ∩ car ∩ car)

Now P(car ∩ car ∩ bus) = $(0.1 \times 0.1 \times 0.6)$

So P(twice by car & once by bus)
$= 3 \times (0.1 \times 0.1 \times 0.6) = 0.018$

P(twice by bicycle & once by car)
= P(bike ∩ bike ∩ car) + P(bike ∩ car ∩ bike)
+ P(car ∩ bike ∩ bike)

Now P(bike ∩ bike ∩ car) = $(0.3 \times 0.3 \times 0.1)$

So P(twice by bicycle & once by car)
$= 3 \times (0.3 \times 0.3 \times 0.1) = 0.027$

Thus,

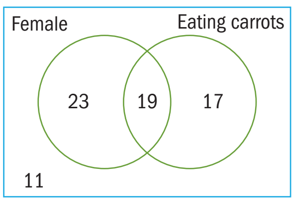
P(twice by car & once by bus or twice by bicycle & once by car) = $0.018 + 0.027 = 0.045$

4 **a** $\frac{6}{16} = \frac{3}{8}$

b $\frac{10}{15} = \frac{2}{3}$

c $\frac{5}{15} \times \frac{4}{14} = \frac{2}{21}$

5



n(Female and not eating carrots) = 23

n(Female and eating carrots) = $42 - 23 = 19$

n(not female and eating carrots) = x

Now $70 - (19 + x) = 34$

$x = 17.$

a P(a rabbit is male and not eating carrots) = $\frac{11}{70}$

b P(a rabbit is female | that it is eating carrots) = $\frac{\frac{19}{70}}{\frac{36}{70}} = \frac{19}{36}$

c No; $P(F) \times P(C) = \frac{42}{70} \times \frac{36}{70} = \frac{78}{2450} \neq P(F \text{ and } C)$

4 Exponential and logarithmic functions

Answers

Skills check

1 **a** $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$

b $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$

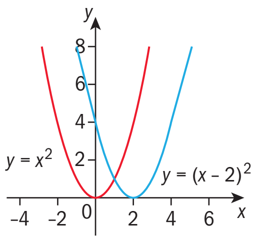
c $0.001^3 = 1\times10^{-9}$

2 **a** $7^n = 343 = 7^3 \Rightarrow n = 3$

b $3^n = 243 = 3^5 \Rightarrow n = 5$

c $5^n = 625 = 5^4 \Rightarrow n = 4$

3



Investigation – folding paper

Number of folds	Number of layers	Thickness (km)	As thick as a
0	1	1×10^{-7}	Piece of paper
1	2	2×10^{-7}	
2	4	4×10^{-7}	Credit card
3	8	8×10^{-7}	
4	16	1.6×10^{-6}	
5	32	3.2×10^{-6}	
6	64	6.4×10^{-6}	
7	128	1.28×10^{-5}	Textbook
8	256	2.56×10^{-5}	
9	512	5.12×10^{-6}	

3 **a** 13 folds

b 15 folds

4 113 000 000 km

Exercise 4A

1 **a** $x^3\times x^2 = x^{3+2} = x^5$

b $3p^2\times 2p^4q^2 = (3\times2)p^{2+4}q^2 = 6p^6q^2$

c $\frac{1}{2}(xy^2)\times\frac{2}{3}(x^2y) = \left(\frac{1}{2}\times\frac{2}{3}\right)x^{1+2}y^{2+1} = \frac{1}{3}x^3y^3$

d $(x^3y^2)(xy^4) = x^{3+1}y^{2+4} = x^4y^6$

2 **a** $x^5\div x^2 = x^{5-2} = x^3$

b $2a^7\div 2a^3 = \frac{2}{2}a^{7-3} = a^4$

c $2a^7\div(2a)^3 = 2a^7\div 8a^3 = \frac{2}{8}a^{7-3} = \frac{a^4}{4}$

d $\frac{4x^3y^5}{2xy^2} = \frac{4}{2}x^{3-1}y^{5-2} = 2x^2y^3$

3 **a** $(x^3)^4 = x^{3\times4} = x^{12}$

b $(3t^2)^3 = 3^3t^{2\times3} = 27t^6$

c $3(x^3y^2)^2 = 3x^{3\times2}y^{2\times2} = 3x^6y^4$

d $(-y^2)^3 = (-1)^3y^{2\times3} = -y^6$

Exercise 4B

1 **a** $9^{\frac{1}{2}} = \sqrt{9} = 3$

b $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

c $64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$

d $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

e $\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{8}{27}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

2 **a** $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

b $32^{-\frac{2}{5}} = \frac{1}{(32)^{\frac{2}{5}}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{4}$

c $81^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{81}} = \frac{1}{3}$

d $\left(2^3\right)^{-\frac{4}{3}} = 2^{3\times-\frac{4}{3}} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

e $\left(\frac{64}{125}\right)^{-\frac{2}{3}} = \frac{1}{\left(\frac{64}{125}\right)^{\frac{2}{3}}} = \left(\frac{125}{64}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{125}}{\sqrt[3]{64}}\right)^2 = \left(\frac{5}{4}\right)^2$

$= \frac{25}{16} = 1\frac{9}{16}$

Exercise 4C

1 **a** $\left(64a^6\right)^{\frac{1}{2}} = \sqrt{64}a^{6\times\frac{1}{2}} = 8a^3$

b $\sqrt[4]{16x^{-8}} = (16x^{-8})^{\frac{1}{4}} = 16^{\frac{1}{4}}x^{\frac{-8}{4}} = \sqrt[4]{16}x^{-8\times\frac{1}{4}} = 2x^{-2} = \frac{2}{x^2}$

c $\frac{q\sqrt{q}}{q^{-1.5}} = \frac{q^{1.5}}{q^{-1.5}} = q^{1.5-(-1.5)} = q^3$

d $\left(\frac{27c^3}{d^3}\right)^{-\frac{1}{3}} = \left(\frac{d^3}{27c^3}\right)^{\frac{1}{3}} = \frac{d^{3\times\frac{1}{3}}}{\sqrt[3]{27}c^{3\times\frac{1}{3}}} = \frac{d}{3c}$

e $\frac{(8p)^{\frac{2}{3}}}{(4p)^2} = \frac{4p^{\frac{2}{3}}}{16p^2} = \frac{1}{4p^{\frac{4}{3}}}$

2 **a** $\dfrac{a^{\frac{3}{2}}}{b^3} \div \dfrac{a^{-1}}{b^2} = \dfrac{a^{\frac{3}{2}}}{b^3} \times \dfrac{b^2}{a^{-1}} = \dfrac{a^{\frac{3}{2}-(-1)}}{b^{3-2}} = \dfrac{a^{\frac{5}{2}}}{b}$

b $\sqrt{\dfrac{x^{-2}y^2}{25x^4}} = \dfrac{x^{-1}y}{5x^2} = \dfrac{y}{5x^3}$ **c** $\dfrac{6x^2y^{-2}}{\sqrt[3]{8x^{-3}}} = \dfrac{6x^2y^{-2}}{2x^{-1}} = \dfrac{3x^3}{y^2}$

Exercise 4D

1 **a** $2^x = 32$

$2^x = 2^5$

$x = 5$

b $3^{1-2x} = 243$

$3^{1-2x} = 3^5$

$1-2x = 5$

$x = -2$

c $3^{x^2-2x} = 27$

$3^{x^2-2x} = 3^3$

$x^2 - 2x = 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, -1$

d $5^{2x-1} - 25 = 0$

$5^{2x-1} = 5^2$

$2x - 1 = 2$

$2x = 3$

$x = \frac{3}{2}$

e $7^{1-x} = \frac{1}{49}$

$7^{1-x} = 7^{-2}$

$1 - x = -2$

$x = 3$

2 **a** $3^{x-3} = 3^{2-x}$

$x - 3 = 2 - x$

$2x = 5$

$x = \frac{5}{2}$

b $5^{3x} = 25^{x-2}$

$5^{3x} = 5^{2(x-2)}$

$3x = 2(x-2)$

$3x = 2x - 4$

$x = -4$

c $9(3^{3x+1}) = \frac{1}{9^x}$

$3^2 \times 3^{3x+1} = (3^{-2})^x$

$3^{3x+3} = 3^{-2x}$

$3x + 3 = -2x$

$5x = -3$

$x = -\frac{3}{5}$

d $2^{2-3x} = 4^{x-1}$

$2^{2-3x} = 2^{2(x-1)}$

$2 - 3x = 2x - 2$

$5x = 4$

$x = \frac{4}{5}$

3 $8(2^{x+1}) = 2\sqrt{2^x}$

$2^3 \times 2^{x+1} = 2 \times 2^{\frac{x}{2}}$

$2^{x+4} = 2^{1+\frac{x}{2}}$

$x + 4 = 1 + \frac{x}{2}$

$2x + 8 = 2 + x$

$x = -6$

Exercise 4E

1 **a** $2x^4 = 162$

$x^4 = 81$

$x^4 = 3^4$

$x = \pm 3$

b $x^5 - 32 = 0$

$x^5 = 32$

$x^5 = 2^5$

$x = 2$

c $x^{-2} = 16$

$1 = 16x^2$

$\frac{1}{16} = x^2$

$x = \pm\frac{1}{4}$

d $8x^{-3} = (8x)^3$

$8x^{-3} = 512x^3$

$\frac{8}{512} = x^6$

$\frac{1}{64} = x^6$

$x = \pm\frac{1}{2}$

e $27x^{-2} = 81x$

$27 = 81x^3$

$\frac{27}{81} = x^3$

$x = \sqrt[3]{\frac{1}{3}}$

f $27x^{-2} = 64$

$27 = 64x^3$

$\frac{27}{64} = x^3$

$x = \frac{3}{4}$

2 **a** $x^{\frac{1}{3}} = 2$

$(x^{\frac{1}{3}})^3 = 2^3$

$x = 8$

b $5x^{\frac{1}{2}} = 125$

$x^{\frac{1}{2}} = 25$

$(x^{\frac{1}{2}})^2 = 25^2$

$x = 625$

c $x^{-\frac{1}{4}} = 4$

$(x^{-\frac{1}{4}})^{-4} = 4^{-4}$

$x = \frac{1}{4^4}$

$x = \frac{1}{256}$

d $x^{\frac{2}{3}} = 16$

$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 16^{\frac{3}{2}}$

$x = \left(\sqrt{16}\right)^3$

$x = 4^3$

$x = 64$

e $x^{-\frac{3}{5}} = \frac{1}{8}$

$\left(x^{-\frac{3}{5}}\right)^{-\frac{5}{3}} = \left(\frac{1}{8}\right)^{-\frac{5}{3}}$

$x = 8^{\frac{5}{3}}$

$x = 2^5$

$x = 32$

f $3x^{-\frac{1}{4}} = 6$

$x^{-\frac{1}{4}} = 2$

$\left(x^{-\frac{1}{4}}\right)^{-4} = 2^{-4}$

$x = \frac{1}{2^4}$

$x = \frac{1}{16}$

3 **a** $x^{-\frac{3}{2}} = 125$

$\left(x^{-\frac{3}{2}}\right)^{-\frac{2}{3}} = 125^{-\frac{2}{3}}$

$x = \frac{1}{125^{\frac{2}{3}}}$

$x = \frac{1}{5^2}$

$x = \frac{1}{25}$

b $6x^{-\frac{2}{3}} = 216$

$x^{-\frac{2}{3}} = 36$

$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = 36$

$x = \frac{1}{36^{\frac{3}{2}}}$

$x = \frac{1}{\left(\sqrt{36}\right)^3}$

$x = \frac{1}{216}$

c $3x^{\frac{2}{3}} = 192$

$x^{\frac{2}{3}} = 64$

$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 64^{\frac{3}{2}}$

$x = \left(\sqrt{64}\right)^3$

$x = 512$

d $9x^{-\frac{2}{3}} = 16$

$x^{-\frac{2}{3}} = \frac{16}{9}$

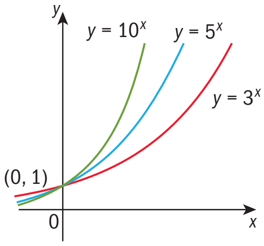
$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = \left(\frac{16}{9}\right)^{-}$

$x = \left(\frac{9}{16}\right)^{\frac{3}{2}}$

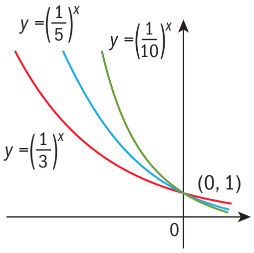
$x = \frac{\left(\sqrt{9}\right)^3}{\left(\sqrt{16}\right)^3}$

$x = \frac{27}{64}$

Investigation – graphs of exponential functions 1



Investigation – graphs of exponential functions 2

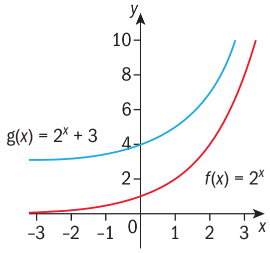


Investigation – compound interest

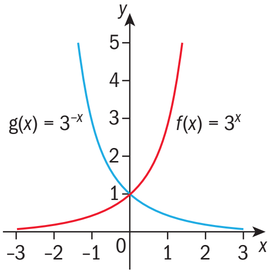
Half-yearly	$\left(1+\frac{1}{2}\right)^2$	2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	2.441 406 25
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	2.613 035 290 22...
Weekly	$\left(1+\frac{1}{52}\right)^{52}$	2.692 596 954 44...
Daily	$\left(1+\frac{1}{365}\right)^{365}$	2.714 567 482 02...
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	2.718 126 690 63...
Every minute	$\left(1+\frac{1}{525\,600}\right)^{525\,60}$	2.718 279 215 4...
Every second	$\left(1+\frac{1}{31\,536\,000}\right)^{31\,536\,000}$	2.718 282 472 54...

Exercise 4F

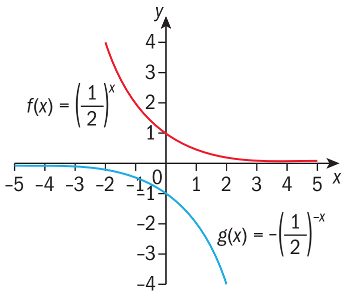
1 a



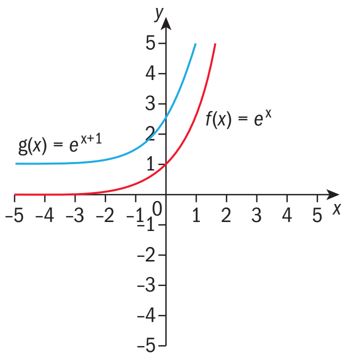
b



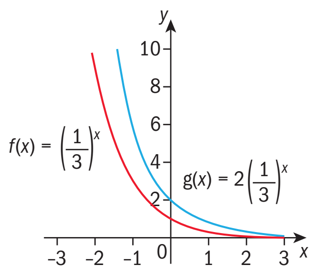
c



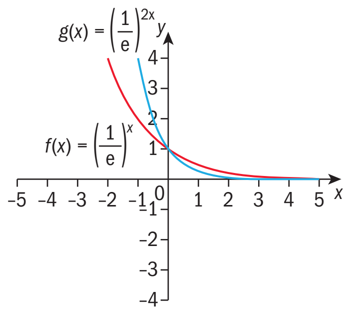
d



e



f



2

	Domain	Range
a	$x \in \mathbb{R}$	$g(x) > 3$
b	$x \in \mathbb{R}$	$g(x) > 0$
c	$x \in \mathbb{R}$	$g(x) < 0$
d	$x \in \mathbb{R}$	$g(x) > 0$
e	$x \in \mathbb{R}$	$g(x) > 0$
f	$x \in \mathbb{R}$	$g(x) > 0$

Exercise 4G

1 **a** $x = \log_7 49$

$7^x = 49$

$7^x = 7^2$

$x = 2$

b $x = \log_5 \sqrt{5}$

$5^x = \sqrt{5}$

$5^x = 5^{\frac{1}{2}}$

$x = \frac{1}{2}$

c $x = \log_2 64$

$2^x = 64$

$2^x = 2^6$

$x = 6$

d $x = \log_9 1$

$9^x = 1$

$9^x = 9^0$

$x = 0$

2 **a** $\log_3 \frac{1}{81} = x$

$3^x = \frac{1}{81}$

$3^x = \frac{1}{3^4}$

$3^x = 3^{-4}$

$x = -4$

b $x = \log_5 125^{\frac{1}{2}}$

$5^x = 125^{\frac{1}{2}}$

$5^x = (5^3)^{\frac{1}{2}}$

$5^x = 5^{\frac{3}{2}}$

$x = \frac{3}{2}$

c $x = \log_{32} 8$

$32^x = 8$

$(2^5)^x = 2^3$

$2^{5x} = 2^3$

$5x = 3$

$x = \frac{3}{5}$

d $x = \log_3 3^4$

$3^x = 3^4$

$x = 4$

Exercise 4H

1 **a** $x = \log_6 6$

$6^x = 6^1$

$x = 1$

b $x = \log_{10} 10$

$10^x = 10^1$

$x = 1$

c $x = \log_n n$

$n^x = n^1$

$x = 1$

d $x = \log_8 1$

$8^x = 1$

$8^x = 8^0$

$x = 0$

e $x = \log_2 1$

$2^x = 1$

$2^x = 2^0$

$x = 0$

f $x = \log_b 1$

$b^x = 1$

$b^x = b^0$

$x = 0$

Exercise 4I

1 **a** $x = 2^9$

$9 = \log_2 x$

b $x = 3^5$

$5 = \log_3 x$

c $x = 10^4$

$4 = \log_{10} x$

d $x = a^b$

$b = \log_a x$

2 **a** $x = \log_2 8$

$2^x = 8$

b $x = \log_3 27$

$3^x = 27$

c $x = \log_{10} 1000$

$10^x = 1000$

d $x = \log_a b$

$a^x = b$

3 **a** $\log_4 x = 3$

$4^3 = x$

$x = 64$

b $\log_3 x = 4$

$3^4 = x$

$x = 81$

c $\log_x 64 = 2$

$x^2 = 64$

$x^2 = 8^2$

$x = 8$

d $\log_x 6 = \frac{1}{2}$

$x^{\frac{1}{2}} = 6$

$x^{\frac{1}{2}} = 36^{\frac{1}{2}}$

$x = 36$

e $\log_2 x = -5$

$2^{-5} = x$

$x = \frac{1}{2^5}$

$x = \frac{1}{32}$

Investigation – inverse functions

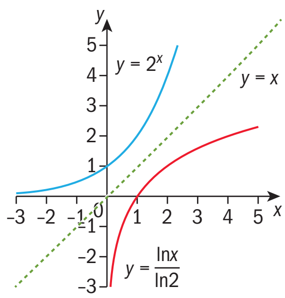
a the function $y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b the inverse function of $y = 2^x$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3

c



d The graphs are reflections of each other in the line $y = x$

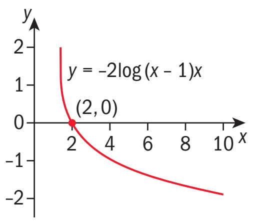
Exercise 4J

1 **a** $f(x)$ is shifted down two units

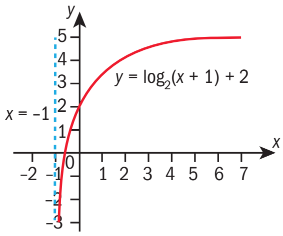
b $f(x)$ is translated right 2 units

c $f(x)$ is stretched by factor 2 parallel with y-axis

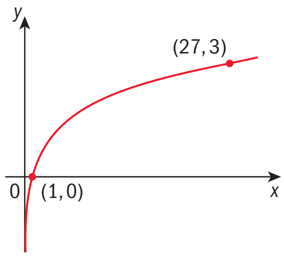
2



3



4



$y = \log_a x$

Sub in point (27,3)

$3 = \log_a 27$

$a^3 = 27$

$a = 3$

5 $f(x) = \log_3 x$

$y = \log_3 x$

Interchange x and y

$x = \log_3 y$

Rearrange to make y the subject

$y = 3^x$

$f^{-1}(x) = 3^x$

$f^{-1}(2) = 3^2 = 9$

Exercise 4K

1 **a** $\log 3 = 0.477$

b $4 \log 2 = 1.20$

c $\ln \sqrt{5} = 0.805$

d $\frac{\log 4}{\log 5} = 0.861$

e $\frac{\ln 4}{\ln 5} = 0.861$

f $\log \frac{4}{5} = -0.0969$

g $(\log 3)^2 = 0.228$

h $\log 3^2 = 0.954$

Exercise 4L

1 **a** $e^x = 1.53$

$x = \ln 1.53$

$x = 0.425$

b $e^x = 0.003$

$x = \ln 0.003$

$x = -5.81$

c $e^x = 1$

$x = \ln 1$

$x = 0$

d $e^x = \frac{1}{2}$

$x = \ln \frac{1}{2}$

$x = -0.693$

e $5e^x = 0.15$

$e^x = 0.03$

$x = \ln 0.03$

$x = -3.51$

2 **a** $10^x = 2.33$

$x = \log 2.33$

$x = 0.367$

b $10^x = 0.6$

$x = \log 0.6$

$x = -0.222$

c $10^x = 1$

$x = \log 1$

$x = 0$

d $10^x = \frac{1}{2}$

$x = \log \frac{1}{2}$

$x = -0.301$

3 **a** $\log x = 2$

$x = 10^2$

$x = 100$

b $\log x = -1$

$x = 10^{-1}$

$x = \frac{1}{10}$

c $\log x = 0$

$x = 10^0$

$x = 1$

d $\log x = -5.1$

$x = 10^{-5.1}$

$x = 0.00000794$

4 **a** $5^{\log_5 12} = 12$

b $5^{\log_5 4} = 4$

c $e^{\ln\sqrt{3}} = \sqrt{3}$

d $e^{\ln 4} = 4$

5 a $\ln e^5 = 5$

b $\log 100 = \log 10^2 = 2$

c $\ln 1 = \ln e^0 = 0$

d $\ln e = 1$

e $\ln\frac{1}{e^3} = \ln e^{-3} = -3$

6 $f(x) = e^{2x-1}$

$y = e^{2x-1}$

$x = e^{2y-1}$

$\ln x = 2y - 1$

$y = \frac{1+\ln x}{2}$

$f^{-1}(x) = \frac{1+\ln x}{2}$

Domain: $x > 0$

7 $f(-2) = e^{-0.5}$, $f(4) = e^{-1}$ since $f(x)$ and $f^{-1}(x)$ are inverses, $f^{-1}(x)$ has domain $[e^{-0.5}, e^{-1}]$ the range of $f(x)$;

and $f^{-1}(x)$ has range $[-2, 4]$ – the domain of $f(x)$.

8 $f(x) = \ln 3x$

$y = \ln 3x$

$x = \ln 3y$

$e^x = 3y$

$y = \frac{1}{3}e^x$

$f^{-1}(x) = \frac{1}{3}e^x$

9 $f(x) = \ln(x-1), x > 1,\ g(x) = 2e^x$

$(g \circ f)(x) = 2e^{\ln(x-1)}$

$= 2(x-1)$

$= 2x - 2$

Exercise 4M

1 a $\log 5 + \log 6$

$= \log(5\times 6) = \log 30$

b $\log 24 - \log 2$

$= \log(24 \div 2) = \log 12$

c $2\log 8 - 4\log 2$

$= \log 8^2 - \log 2^4$

$= \log 64 - \log 16$

$= \log\left(\frac{64}{16}\right)$

$= \log 4$

d $\frac{1}{2}\log 49$

$= \log 49^{\frac{1}{2}}$

$= \log 7$

e $3\log x - 2\log y$

$= \log x^3 - \log y^2$

$= \log\frac{x^3}{y^2}$

f $\log x - \log y - \log z$

$= \log x - (\log y + \log z)$

$= \log\frac{x}{yz}$

g $\log x + 2\log y - 3\log xy$

$= \log x + \log y^2 - \log x^3y^3$

$= \log\frac{xy^2}{x^3y^3}$

$= \log\frac{1}{x^2y}$

2 a $\log_2 6 + 2\log_2 3 - \log_2 4$

$= \log_2 6 + \log_2 9 - \log_2 4$

$= \log_2\left(\frac{6\times 9}{4}\right)$

$= \log_2\left(\frac{27}{2}\right)$

b $\log_3 40 - \log_3 15 + 2\log_3\left(\frac{3}{5}\right)$

$= \log_3 40 - \log_3 15 + \log_3\frac{9}{25}$

$= \log_3\left(\frac{40}{15}\times\frac{9}{25}\right)$

$= \log_3\left(\frac{24}{25}\right)$

c $\log_a 4 + 2\log_a 3 - \log_a 6$

$= \log_a 4 + \log_a 9 - \log_a 6$

$= \log_a\left(\frac{36}{6}\right)$

$= \log_a 6$

d $2\ln 3 - \ln 18 = \ln 9 - \ln 18 = \ln\left(\frac{1}{2}\right)$ or $-\ln 2$

e $3\ln 2 - 2 = \ln 8 - \ln e^2 = \ln\left(\frac{8}{e^2}\right)$

f $4\log_2 x + \frac{1}{3}\log_2 y - 5\log_2 z$

$= \log_2 x^4 + \log_2 y^{\frac{1}{3}} - \log_2 z^5$

$= \log_2\left(\frac{x^4y^{\frac{1}{3}}}{z^5}\right)$

3 a $\log_6 2 + \log_6 18 = \log_6 36 = \log_6 6^2 = 2$

b $\log_2 24 - \log_2 3 = \log_2 8 = \log_2 2^{3n} = 3$

c $\log_8 2 + \log_8 32 = \log_8 64 = \log_8 8^2 = 2$

d $2\log_6 3 + \log_6 24 = \log_6(9 \times 24)$
$= \log_6 216 = \log_6 6^3 = 3$

e $\frac{1}{2}\log 36 - \log 15 + 2\log 5$
$= \log 6 - \log 15 + \log 25$
$= \log\left(\frac{6}{15} \times 25\right) = \log 10 = 1$

Exercise 4N

1 **a** $\log_2 ab = \log_2 a + \log_2 b = p + q$

b $\log_2 a^3 = 3\log_2 a = 3p$

c $\log_2 \frac{b}{a} = \log_2 b - \log_2 a = q - p$

d $\log_2 \sqrt{b} = \frac{1}{2}\log_2 b = \frac{q}{2}$

e $\log_2 \frac{b^2}{\sqrt{a}} = \log_2 b^2 - \log_2 a^{\frac{1}{2}} = 2\log_2 b - \frac{1}{2}\log_2 a = 2q - \frac{p}{2}$

2 **a** $\log\left(\frac{P^2}{QR^2}\right)^3 = 3\log\left(\frac{P^2}{QR^2}\right)$
$= 3\left[\log P^2 - \log\left(QR^2\right)\right]$
$= 3\left[2\log P - \left(\log Q + \log R^2\right)\right]$
$= 3[2\log P - \log Q - 2\log R]$
$= 6\log P - 3\log Q - 6\log R$
$= 6x - 3y - 6z$

3 **a** $\log 10x = \log 10 + \log x = 1 + \log x$

b $\log\frac{100}{x^2} = \log 100 - \log x^2 = 2 - 2\log x$

c $\log\sqrt{10x} = \frac{1}{2}\log 10x = \frac{1}{2}(\log 10 + \log x) = \frac{1}{2} + \frac{1}{2}\log x$

d $\log\frac{1}{10\sqrt{x}} = \log 1 - \log 10x^{\frac{1}{2}} = -[\log 10 + \frac{1}{2}\log x]$
$= -1 - \frac{1}{2}\log x$

4 $y = \log_3 \frac{27^a}{81}$
$y = \log_3 27^a - \log_3 81$
$y = a\log_3 27 - \log_3 81$
$y = 3a - 4$

5 $\log_3 \frac{1}{27x^2}$
$= \log_3 1 - [\log_3 27 + \log_3 x^2]$
$= -[\log_3 27 + 2\log_3 x]$
$= -3 - 2\log_3 x$

6 $e^{x\ln 2} = e^{\ln 2^x} = 2^x$

Exercise 4O

1 **a** $\log_2 7 = \frac{\log 7}{\log 2} = 2.81$

b $\log_5\left(\frac{1}{7}\right) = \frac{\log\left(\frac{1}{7}\right)}{\log 5} = -1.21$

c $\log_3(0.7) = \frac{\log 0.7}{\log 3} = -0.325$

d $\log_7 e = \frac{\ln e}{\ln 7} = \frac{1}{\ln 7} = 0.514$

e $\log_3 7^7 = 7\frac{\log 7}{\log 3} = 12.4$

2 $\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{y}{2}$

3 **a** $\log_2 6 = \frac{\log_a 6}{\log_a 2} = \frac{y}{x}$

b $\log_6 2 = \frac{\log_a 2}{\log_a 6} = \frac{x}{y}$

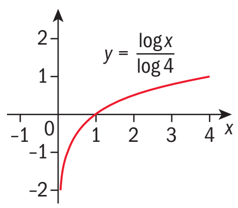
c $\log_2 36 = \frac{\log_a 36}{\log_a 2} = \frac{\log_a 6^2}{\log_a 2} = \frac{2\log_a 6}{\log_a 2} = \frac{2y}{x}$

d $\log_a 24 = \log_a(6 \times 4) = \log_a 6 + \log_a 4$
$= \log_a 6 + 2\log_a 2$
$= 2x + y$

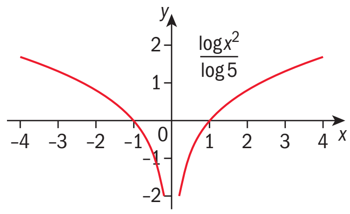
e $\log_6 12 = \frac{\log_a 12}{\log_a 6} = \frac{\log_a 2 + \log_a 6}{\log_a 6} = \frac{x + y}{y}$

f $\log_2 3 = \frac{\log_a 3}{\log_a 2} = \frac{\log_a\left(\frac{6}{2}\right)}{\log_a 2}$
$= \frac{\log_a 6 - \log_a 2}{\log_a 2}$
$= \frac{y - x}{x}$

4 **a**



b



5 **a** $y = \log_4 a^2$
$y = 2\log_4 a$
$y = 2b$

b $y = \log_{16} a$
$y = \frac{\log_4 a}{\log_4 16}$
$y = \frac{b}{2}$

c $y=\log_{\frac{1}{4}} a^2$

$y=\dfrac{\log_4 a^2}{\log_4 \frac{1}{4}}$

$y=\dfrac{2\log_4 a}{-1}$

$y=-2b$

d $y=\log_{\frac{1}{16}} \sqrt{a}$

$y=\dfrac{\log_4 \sqrt{a}}{\log_4 \frac{1}{16}}$

$y=\dfrac{\frac{1}{2}\log_4 a}{-2}$

$y=-\dfrac{b}{4}$

Exercise 4P

1 **a** $2^x=5$

$\log 2^x=\log 5$

$x\log 2=\log 5$

$x=\dfrac{\log 5}{\log 2}$

$x=2.32$

b $3^x=50$

$\log 3^x=\log 50$

$x\log 3=\log 50$

$x=\dfrac{\log 50}{\log 3}$

$x=3.56$

c $5^{-x}=17$

$\log 5^{-x}=\log 17$

$-x\log 5=\log 17$

$-x=\dfrac{\log 17}{\log 5}$

$x=-1.76$

d $7^{x+1}=16$

$(x+1)\log 7=\log 16$

$x\log 7+\log 7=\log 16$

$x\log 7=\log 16-\log 7$

$x=\dfrac{\log 16-\log 7}{\log 7}$

$x=0.425$

e $\left(\dfrac{1}{3}\right)^x=\dfrac{7}{9}$

$x\log\left(\dfrac{1}{3}\right)=\log\dfrac{7}{9}$

$x=\dfrac{\log\frac{7}{9}}{\log\frac{1}{3}}$

$x=0.229$

f $2^{2x-1}=3.2\times 10^{-3}$

$(2x-1)\log 2=\log(3.2\times 10^{-3})$

$2x\log 2-\log 2=\log(3.2\times 10^{-3})$

$2x\log 2=\log(3.2\times 10^{-3})+\log 2$

$x=\dfrac{\log(3.2\times 10^{-3})+\log 2}{2\log 2}$

$x=-3.64$

g $e^x=6$

$\ln e^x=\ln 6$

$x=1.79$

h $e^{\frac{x}{5}}=0.11$

$\ln e^{\frac{x}{5}}=\ln 0.11$

$\dfrac{x}{5}=\ln 0.11$

$x=5\ln 0.11$

$x=-11.0$

2 **a** $2^{x+2}=5^{x-3}$

$(x+2)\log 2=(x-3)\log 5$

$x\log 2+2\log 2=x\log 5-3\log 5$

$2\log 2+3\log 5=x\log 5-x\log 2$

$x(\log 5-\log 2)=(2\log 2+3\log 5)$

$x=\dfrac{(2\log 2+3\log 5)}{(\log 5-\log 2)}$

$x=6.78$

b $3^{2-x}=4^{2x-5}$

$(2-x)\log 3=(2x-5)\log 4$

$2\log 3-x\log 3=2x\log 4-5\log 4$

$2\log 3+5\log 4=2x\log 4+x\log 3$

$x(2\log 4+\log 3)=(2\log 3+5\log 4)$

$x=\dfrac{(2\log 3+5\log 4)}{(2\log 4+\log 3)}$

$x=2.36$

c $3^{\frac{x}{3}}=5^{x+3}$

$\dfrac{x}{3}\log 3=(x+3)\log 5$

$\dfrac{x}{3}\log 3=x\log 5+3\log 5$

$\dfrac{x}{3}\log 3-x\log 5=3\log 5$

$x\left(\dfrac{1}{3}\log 3-\log 5\right)=3\log 5$

$x=\dfrac{3\log 5}{\left(\frac{1}{3}\log 3-\log 5\right)}$

$x=-3.88$

d $7^x = (0.5)^{x-1}$

$x\log 7 = (x-1)\log 0.5$

$x\log 7 = x\log 0.5 - \log 0.5$

$\log 0.5 = x\log 0.5 - x\log 7$

$x(\log 0.5 - \log 7) = \log 0.5$

$x = \frac{\log 0.5}{(\log 0.5 - \log 7)}$

$x = 0.263$

e $e^{3x-1} = 3^x$

$(3x-1)\ln e = x\ln 3$

$3x-1 = x\ln 3$

$3x - x\ln 3 = 1$

$x(3-\ln 3) = 1$

$x = \frac{1}{(3-\ln 3)}$

$x = 0.526$

f $4e^{3x-2} = 244$

$e^{3x-2} = 61$

$(3x-2)\ln e = \ln 61$

$3x-2 = \ln 61$

$3x = \ln 61 + 2$

$x = \frac{\ln 61 + 2}{3}$

$x = 2.04$

g $35e^{-0.01x} = 95$

$e^{-0.01x} = \frac{19}{7}$

$(-0.01x)\ln e = \ln\frac{19}{7}$

$-0.01x = \ln\frac{19}{7}$

$x = \ln\frac{19}{7} \div (-0.01)$

$x = -99.9$

Exercise 4Q

1 **a** $7\times 3^x = 25$

$3^x = \frac{25}{7}$

$x\log 3 = \log\frac{25}{7}$

$x = \frac{\log\frac{25}{7}}{\log 3}$

$x = 1.16$

b $4\times 3^x = 5^{2x-1}$

$\log 4 + \log 3^x = \log 5^{2x-1}$

$\log 4 + x\log 3 = (2x-1)\log 5$

$\log 4 + x\log 3 = 2x\log 5 - \log 5$

$\log 4 + \log 5 = 2x\log 5 - x\log 3$

$x(2\log 5 - \log 3) = (\log 4 + \log 5)$

$x = \frac{(\log 4 + \log 5)}{(2\log 5 - \log 3)}$

$x = 1.41$

c $3\times 2^x = 4\times 5^x$

$\log 3 + \log 2^x = \log 4 + \log 5^x$

$\log 3 + x\log 2 = \log 5 + x\log 5$

$x\log 2 - x\log 5 = \log 5 - \log 3$

$x(\log 2 - \log 5) = \log 5 - \log 3$

$x = \frac{(\log 5 - \log 3)}{(\log 2 - \log 5)}$

$x = -0.557$

d $5\times 2^{x-1} = 3\times 7^{2x}$

$\log 5 + (x-1)\log 2 = \log 3 + 2x\log 7$

$\log 5 + x\log 2 - \log 2 = \log 3 + 2x\log 7$

$\log 5 - \log 2 - \log 3 = 2x\log 7 - x\log 2$

$x(2\log 7 - \log 2) = \log 5 - \log 2 - \log 3$

$x = \frac{(\log 5 - \log 2 - \log 3)}{(2\log 7 - \log 2)}$

$x = -0.0570$

e $3^x 4^{x-1} = 2\times 7^{x+2}$

$\log 3^x + \log 4^{x-1} = \log 2 + \log 7^{x+2}$

$x\log 3 + (x-1)\log 4 = \log 2 + (x+2)\log 7$

$x\log 3 + x\log 4 - \log 4 = \log 2 + x\log 7 + 2\log 7$

$x(\log 3 + \log 4 - \log 7) = \log 2 + 2\log 7 + \log 4$

$x = \frac{(\log 2 + 2\log 7 + \log 4)}{(\log 3 + \log 4 - \log 7)}$

$x = 11.1$

2 **a** $2^{x+2} = 5^{x-3}$

$(x+2)\ln 2 = (x-3)\ln 5$

$x\ln 2 + 2\ln 2 = x\ln 5 - 3\ln 5$

$2\ln 2 + 3\ln 5 = x\ln 5 - x\ln 2$

$\ln 4 + \ln 125 = x(\ln 5 - \ln 2)$

$x\ln\frac{5}{2} = \ln 500$

$x = \frac{\ln 500}{\ln\frac{5}{2}}$

b $5 \times 3^x = 8 \times 7^x$

$\ln 5 + \ln 3^x = \ln 8 + \ln 7^x$

$\ln 5 + x\ln 3 = \ln 8 + x\ln 7$

$x\ln 3 - x\ln 7 = \ln 8 - \ln 5$

$x(\ln 3 - \ln 7) = \ln 8 - \ln 5$

$x = \frac{\ln\frac{8}{5}}{\ln\frac{3}{7}}$

c $5 \times 3^{x+1} = 2 \times 6^{3-2x}$

$\ln 5 + (x+1)\ln 3 = \ln 2 + (3-2x)\ln 6$

$\ln 5 + x\ln 3 + \ln 3 = \ln 2 + 3\ln 6 - 2x\ln 6$

$x\ln 3 + 2x\ln 6 = \ln 2 + 3\ln 6 - \ln 5 - \ln 3$

$x(\ln 3 + 2\ln 6) = \ln 2 + \ln 216 - \ln 5 - \ln 3$

$x\ln 108 = \ln\frac{144}{5}$

$x = \frac{\ln\frac{144}{5}}{\ln 108}$

d $(6^x)(2^{x-1}) = 2(4^{x+2})$

$x\ln 6 + (x-1)\ln 2 = \ln 2 + (x+2)\ln 4$

$x\ln 6 + x\ln 2 - \ln 2 = \ln 2 + x\ln 4 + 2\ln 4$

$x\ln 6 + x\ln 2 - x\ln 4 = \ln 2 + 2\ln 4 + \ln 2$

$x(\ln 6 + \ln 2 - \ln 4) = \ln 2 + \ln 16 + \ln 2$

$x\ln 3 = \ln 64$

$x = \frac{\ln 64}{\ln 3}$

3 a $e^{2x} - 2e^x = 0$

$e^{2x} = 2e^x$

$\ln e^{2x} = \ln 2 + \ln e^x$

$2x = \ln 2 + x$

$x = \ln 2$

b $4^x - 3(2^x) = 0$

$4^x = 3(2^x)$

$x\ln 4 = \ln 3 + x\ln 2$

$x(\ln 4 - \ln 2) = \ln 3$

$x = \frac{\ln 3}{\ln 2}$

Exercise 4R

1 a $\log_2(x) = \log_2(6x-1)$

$x = 6x - 1$

$5x = 1$

$x = \frac{1}{5}$

b $\ln(x+1) = \ln(3-x)$

$x + 1 = 3 - x$

$2x = 2$

$x = 1$

c $\log_5(2-x) = \log_5(6x-1)$

$2 - x = 6x - 1$

$3 = 7x$

$x = \frac{3}{7}$

d $\log_2(2x+3) + \log_2(x-1) = \log_2(x+1)$

$\log_2[(2x+3)(x-1)] = \log_2(x+1)$

$2x^2 + 3x - 2x - 3 = x + 1$

$2x^2 = 4$

$x^2 = 2$

$x = \sqrt{2}$

We cannot have $x = -\sqrt{2}$, since then $\log_2(x-1)$ would be undefined.

e

$\log_3 x - \log_3(x-1) = \log_3(x+1)$

$\log_3\left(\frac{x}{x-1}\right) = \log_3(x+1)$

$\left(\frac{x}{x-1}\right) = x + 1$

$x = (x+1)(x-1)$

$x = x^2 - 1$

$0 = x^2 - x - 1$

$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

$x = \frac{1+\sqrt{5}}{2}$ since we can't have $\log_3 x$ when $x < 0$.

$x = 1.62$

Exercise 4S

1 a $\log_9(x-2) = 2$

$x - 2 = 9^2$

$x = 81 + 2 = 83$

b $\log_3(2x-1) = 3$

$2x - 1 = 3^3$

$2x - 1 = 27$

$2x = 28$

$x = 14$

c $\log_{\frac{1}{2}}(3-x) = 5$

$3 - x = \left(\frac{1}{2}\right)^5$

$3 - x = \frac{1}{32}$

$x = \frac{95}{32}$

2 a $\log_6(x-5) + \log_6 x = 2$

$\log_6[x(x-5)] = 2$

$x^2 - 5x = 6^2$

$x^2 - 5x - 36 = 0$

$(x-9)(x+4) = 0$

$x = 9$

We can't have $x = -4$, since then $\log_6 x$ is undefined.

b $\log_2(4x-8)-\log_2(x-5)=4$

$\log_2\frac{4x-8}{x-5}=4$

$\frac{4x-8}{x-5}=2^4$

$4x-8=16(x-5)$

$4x-8=16x-80$

$72=12x$

$x=6$

c $\log_7(2x-3)-\log_7(4x-5)=0$

$\log_7(2x-3)=\log_7(4x-5)$

$2x-3=4x-5$

$2x=2$

$x=1$

When $x=1$, $(2x-3)<0$ and $(4x-5)<0$, so there are no solutions.

3 $\log_2 x + \log_2(2x+7)=\log_2 A$

$\log_2[x(2x+7)]=\log_2 A$

$2x^2+7x=A$

When $\log_2 A=2$, then $A=2^2=4$

$2x^2+7x=4$

$2x^2+7x-4=0$

$(2x-1)(x+4)=0$

$x=0.5$

4 $\log_4 x+\log_x 4=2$

$\log_4 x+\frac{\log_4 4}{\log_4 x}=2$

$(\log_4 x)^2+\log_4 4=2\log_4 x$

$(\log_4 x)^2-2\log_4 x+1=0$

Let $a=\log_4 x$. Then

$a^2-2a+1=0$

$(a-1)^2=0$

$a=1$

so $\log_4 x=1$

$\therefore\ x=4$

5 $\log_2 x^2+\log_4\sqrt{x}=9$

$\log_2 x^2+\frac{\log_2 x^{\frac{1}{2}}}{\log_2 4}=9$

$2\log_2 x+\frac{\log_2 x^{\frac{1}{2}}}{2}=9$

$2\log_2 x+\frac{\log_2 x}{4}=9$

$8\log_2 x+\log_2 x=36$

$9\log_2 x=36$

$\log_2 x=4$

$x=16$

Exercise 4T

1 **a** 450×1.032^n

b $450\times1.032^n>600$

$1.032^n>\frac{4}{3}$

$n\log1.032>\log\frac{4}{3}$

$n>\frac{\log\frac{4}{3}}{\log1.032}$

$n>9.133..$

$n=10$ years

2 **a** **i** $100\times1.1^2=121$

ii $100\times1.1^7=195$

b $100\times1.1^n=250$

$1.1^n=2.5$

$n=\frac{\log2.5}{\log1.1}$

$=9.6$ days (10 days)

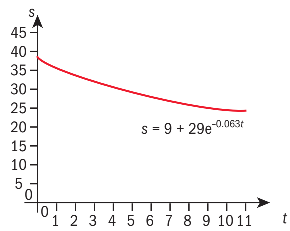
3 $10\times1.15^n=10000$

$n\log1.15=\log1000$

$n=\frac{\log1000}{\log1.15}$

49.4 hours

4 **a**



b $9+29e^{-0.063(0)}=38\text{ ms}^{-1}$

c as $t\to\infty$, $s\to9\text{ ms}^{-1}$

d $9+29e^{-0.063(45)}=10.7\text{ ms}^{-1}$

e His initial speed was 38 ms^{-1}. We need to find the time at which $s=19\text{ ms}^{-1}$.

$9+29e^{-0.063t}=19$

$29e^{-0.063t}=10$

$e^{-0.063t}=\frac{10}{29}$

$-0.063t=\ln\frac{10}{29}$

$t=16.9...$

$t=17$ sec

5 $x=a\times n^b$

$32=a\times2^b$ (1)

$108=a\times3^b$ (2)

$(1)\Rightarrow\ \ln32=\ln a+b\ln2$

$(2)\Rightarrow\ \ln108=\ln a+b\ln3$

Solve simultaneously

(2)−(1) $\ln 108 - \ln 32 = b\ln 3 - b\ln 2$

$\ln 108 - \ln 32 = b(\ln 3 - \ln 2)$

$b = \frac{\ln 108 - \ln 32}{\ln 3 - \ln 2}$

$b = 3,\ a = \frac{32}{2^3} = 4$

Review exercise

1 $x = \log_5 287$

$x = \frac{\log 287}{\log 5} = 3.52$

2 **a** $3^{2x+3} = 90$

$(2x+3)\log 3 = \log 90$

$2x\log 3 + 3\log 3 = \log 90$

$2x\log 3 = \log 90 - 3\log 3$

$x = \frac{\log 90 - 3\log 3}{2\log 3}$

$= 0.548$

b $5^{x-1} = 3^{3x}$

$(x-1)\log 5 = 3x\log 3$

$x\log 5 - \log 5 = 3x\log 3$

$x(\log 5 - 3\log 3) = \log 5$

$x = \frac{\log 5}{\log 5 - 3\log 3}$

$x = -0.954$

c $2 \times 3^{2x} = 5^x$

$\log 2 + 2x\log 3 = x\log 5$

$\log 2 = x(\log 5 - 2\log 3)$

$x = \frac{\log 2}{\log 5 - 2\log 3}$

$x = -1.18$

3 **a** $\log x + \log(3x - 13) = 1$

$\log[x(3x - 13)] = 1$

$3x^2 - 13x = 10^1$

$3x^2 - 13x - 10 = 0$

$(3x+2)(x-5) = 0$

$x = 5$

b $\log_5(x+6) - \log_5(x+2) = \log_5 x$

$\log_5 \frac{x+6}{x+2} = \log_5 x$

$\frac{x+6}{x+2} = x$

$x + 6 = x^2 + 2x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = 2$

c $\ln(4x-7) = 2$

$4x - 7 = e^2$

$4x = e^2 + 7$

$x = \frac{e^2 + 7}{4}$

$x = 3.60$

d $\log_2(x^2) = (\log_2 x)^2$

$2\log_2 x = (\log_2 x)^2$

$(\log_2 x)^2 - 2\log_2 x = 0$

$\log_2 x(\log_2 x - 2) = 0$

$\log_2 x = 0, \log_2 x = 2$

$x = 1$ and 4

e $\log_{10} x = 4\log_x 10$

$\log x = 4\frac{\log 10}{\log x}$

$(\log x)^2 = 4$

$\log x = 2, -2$

$x = 100, \frac{1}{100}$

4 **a** $f(x) > 0$, range of $g(x)$ is all real numbers

b Inverses as 1-1 functions;

$f(x) = e^{2x}$

$y = e^{2x}$

$x = e^{2y}$

$\ln x = 2y$

$y = \frac{1}{2}\ln x$

$f^{-1}(x) = \frac{1}{2}\ln x$

$g(x) = \frac{3}{2}\ln x$

$y = \frac{3}{2}\ln x$

$x = \frac{3}{2}\ln y$

$\frac{2}{3}x = \ln y$

$y = e^{\frac{2}{3}x}$

$g^{-1}(x) = e^{\frac{2}{3}x}$

c $(f \circ g)(x) = e^{2\left(\frac{3}{2}\ln x\right)} = e^{3\ln x} = e^{\ln x^3}$

$= x^3$

$(g \circ f)(x) = \frac{3}{2}\ln(e^{2x}) = \frac{3}{2}(2x)$

$= 3x$

d $x^3 = 3x$

$x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$x = \sqrt{3}$

5 **a** $n = 4000e^{0.08t}$

$n = 4000e^{0.08(50)} = 218\ 393$ insects

b $t = 0$; $n = 4000e^{0.08(0)} = 4000$

We want to find the time at which the population reaches 8000.

$4000e^{0.08(t)} = 8000$

$e^{0.08(t)} = 2$

$0.08t = \ln 2$

$t = \frac{\ln 2}{0.08} = 8.66$

8.66 days.

Review exercise

1 $25^{4x-3} = \left(\frac{1}{125}\right)^{x+1}$

$(5^2)^{4x-3} = (5^{-3})^{x+1}$

$5^{8x-6} = 5^{-3x-3}$

$8x - 6 = -3x - 3$

$11x = 3$

$x = \frac{3}{11}$

2 $(5^{x+1})(7^x) = 3^{2x+1}$

$\log 5^{x+1} + \log 7^x = \log 3^{2x+1}$

$(x+1)\log 5 + x\log 7 = (2x+1)\log 3$

$x(\log 5 + \log 7 - 2\log 3) = \log 3 - \log 5$

$x = \frac{\log\left(\frac{3}{5}\right)}{\log\left(\frac{5\times 7}{3^2}\right)}$

$x = \frac{\log\left(\frac{3}{5}\right)}{\log\left(\frac{35}{9}\right)}$

3 $2\log_3 27 + \log_3\left(\frac{1}{3}\right) - \log_3\sqrt{3}$

$= 2\log_3(3^3) + \log_3(3^{-1}) - \log_3\left(3^{\frac{1}{2}}\right)$

$= 2(3) + (-1) - (0.5) = 4.5$

4 $4\log_3 x + \frac{1}{3}\log_3 y - 5\log_3 z$

$= \log_3 x^4 + \log_3 \sqrt[3]{y} - \log_3 z^5$

$= \log_3 \frac{x^4\sqrt[3]{y}}{z^5}$

5 **a** $\log_3(4x-1) = 3$

$4x - 1 = 3^3$

$4x = 28$

$x = 7$

b $\log_x(2x-1) = 2$

$x^2 = 2x - 1$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

c $\log_x(5x-4) = 2$

$x^2 = 5x - 4$

$x^2 - 5x + 4 = 0$

$(x-4)(x+1) = 0$

$x = 1, 4$

d $\log_2(x-2) + \log_{\frac{1}{2}}(x-1) = 3$

$\log_2(x-2) + \frac{\log_2(x-1)}{\log_2\frac{1}{2}} = 3$

$\log_2(x-2) - \log_2(x-1) = 3$

$\log_2\frac{x-2}{x-1} = 3$

$\frac{x-2}{x-1} = 8$

$x - 2 = 8x - 8$

$7x = 6$

$x = \frac{6}{7}$

6 **a** $\log_4 8 = \frac{\log_x 8}{\log_x 4} = \frac{n}{m}$

b $\log_x 2 = \log_x 8 - \log_x 4 = n - m$

c $\log_x 16 = \log_x 4^2 = 2\log_x 4 = 2m$

d $\log_8 32 = \frac{\log_x 32}{\log_x 8} = \frac{\log_x 4 + \log_x 8}{\log_x 8} = \frac{m+n}{n}$

7 Shift one unit to the right, stretch factor $\frac{1}{3}$ parallel to x-axis, shift 2 units up.

8 **a** $f(x) = 3e^{2x}$

$y = 3e^{2x}$

$x = 3e^{2y}$

$\frac{x}{3} = e^{2y}$

$\ln\frac{x}{3} = 2y$

$y = \frac{1}{2}\ln\left(\frac{x}{3}\right)$

$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x}{3}\right)$

b $f(x) = 10^{3x}$

$y = 10^{3x}$

$x = 10^{3y}$

$\log x = \log 10^{3y}$

$3y = \log x$

$y = \frac{1}{3}\log x$

$f^{-1}(x) = \frac{1}{3}\log x$

c $f(x) = \log_2(4x)$

$y = \log_2 4x$

$x = \log_2 4y$

$2^x = 4y$

$y = \frac{2^x}{4}$

$f^{-1}(x) = \frac{2^x}{4} = 2^{x-2}$

9 $\log_a 64 + \log_a b = 8$ ①

$\log_b a = \frac{1}{2}$ ②

from ②

$a = b^{\frac{1}{2}} \Rightarrow b = a^2$

sub in ①

$\log_a 64 + \log_a a^2 = 8$

$\log_a 64 + 2 = 8$

$\log_a 64 = 6$

$a^6 = 64$

$a^6 = 2^6$

$a = 2$

from ②

$b = a^2 = 4$

$a = 2, b = 4$

5 Rational functions

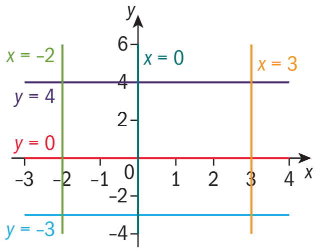
Answers

Skills check

1 **a** $-8x+20$ **b** $12x-18$ **c** $-x^3-7x$

d $x^4+6x^3+9x^2$ **e** x^3+5x^2-24x

2



3 The parent function is $y=x^3$

A is a horizontal shift of 4 units to the right.

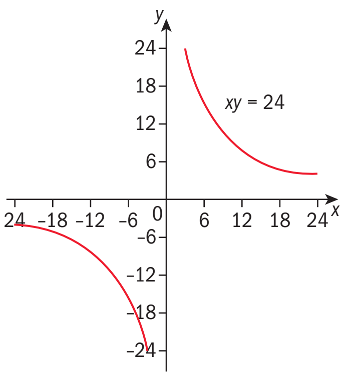
Function A is $y=(x-4)^3$.

B is a vertical shift of 2 units down.

Function B is $y=x^3-2$.

Investigation - graphing product pairs

x	24	12	8	3	6	4	2	1
y	1	2	3	8	4	6	12	24



As y gets bigger, x gets smaller and vice versa.
The graph gets closer and closer to the axes as x- and y-values increase.

Exercise 5A

1 **a** $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $-\frac{1}{3}$ **d** -1 **e** $\frac{3}{2}$ **f** $\frac{11}{7}$

g $-\frac{2}{3}$ **h** $3\frac{1}{2}=\frac{7}{2}$, the reciprocal is $\frac{2}{7}$

2 **a** $6.5=\frac{13}{2}$, the reciprocal is $\frac{2}{13}$

b $\frac{1}{x}$ **c** $\frac{1}{y}$ **d** $\frac{1}{3x}$ **e** $\frac{1}{4y}$ **f** $\frac{9}{2x}$ **g** $\frac{5}{3a}$ **h** $\frac{3d}{2}$

i $\frac{t}{d}$ **j** $\frac{x-1}{x+1}$

3 **a** $6\times\frac{1}{6}=1$ **b** $\frac{3}{4}\times\frac{4}{3}=1$ **c** $\frac{2c}{3d}\times\frac{3d}{2c}=1$

4 **a** $\frac{1}{\frac{1}{4}}=1\times\frac{4}{1}=4$ **b** $\frac{1}{\frac{1}{x}}=1\times\frac{x}{1}=x$

5 **a** **i** $48y=24$

$y=\frac{24}{48}=0.5$

ii $480y=24$

$y=\frac{24}{480}=0.05$

iii $4800y=24$

$y=\frac{24}{4800}=0.005$

iv $48\,000y=24$

$y=\frac{24}{48\,000}=0.0005$

b As x gets larger, y gets smaller, nearer to zero.

c y will never reach zero, as $y=\frac{24}{x}>0$ for all $x>0$

d **i** $48x=24$

$x=\frac{24}{48}=0.5$

ii $480x=24$

$x=\frac{24}{480}=0.05$

iii $4800x=24$

$x=\frac{24}{480}=0.005$

iv $48000x=24$

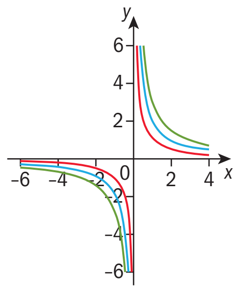
$x=\frac{24}{4800}=0.0005$

e As y gets larger, x gets smaller, nearer to zero

f x will never reach zero, as $x=\frac{24}{y}>0$ for all $y>0$.

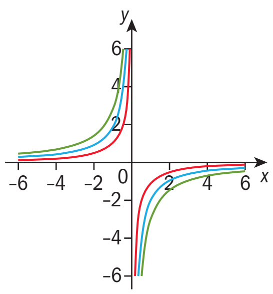
Investigation — graphs of reciprocal functions

1 **a** $f(x)=\frac{1}{x}$ **b** $g(x)=\frac{2}{x}$ **c** $h(x)=\frac{3}{x}$



The numerator indicates the scale factor of the vertical stretch.

2



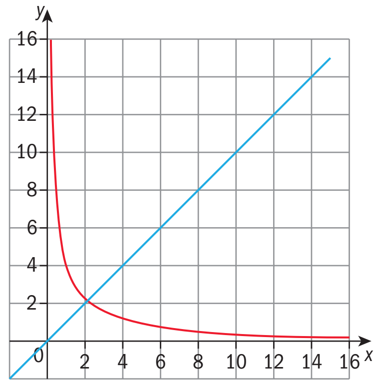
Changing the sign of the numerator reflects the graphs of the original functions over the *x*-axis.

3

x	0.25	0.4	0.5	1	2	4	8	10	16
$f(x)$	16	10	8	4	2	1	0.5	0.4	0.25

b The top and bottom rows of the table contain the same numbers but in reverse order.

c d e

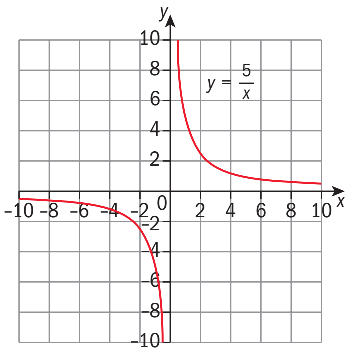


f The function reflects onto itself.

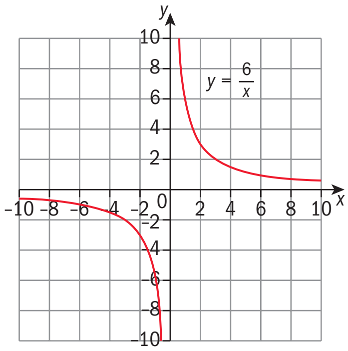
g The function is its own inverse.

Exercise 5B

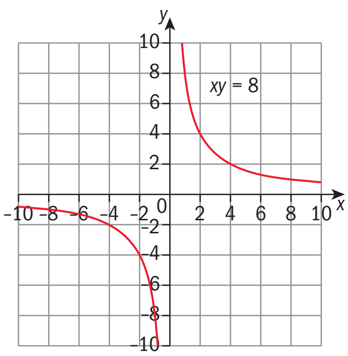
1 a



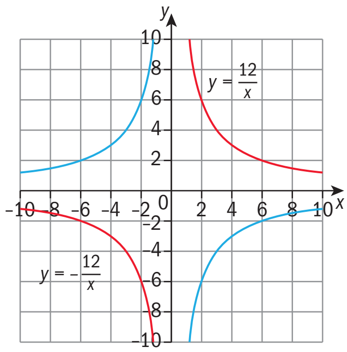
b



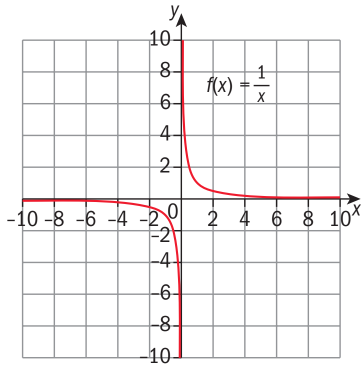
c



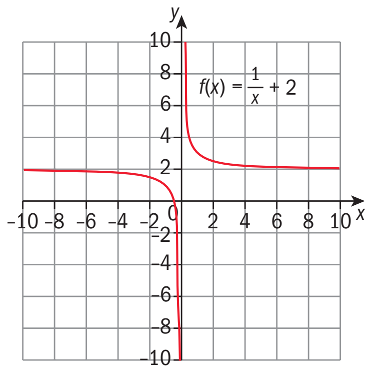
2



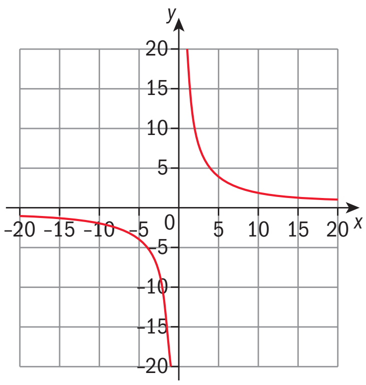
3 a $x = 0$ and $y = 0$



b $x = 0$ and $y = 2$



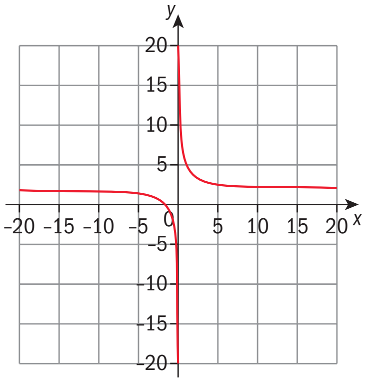
4 a



$y = 0$, $x = 0$. Domain $x \in \mathbb{R}$, $x \neq 0$

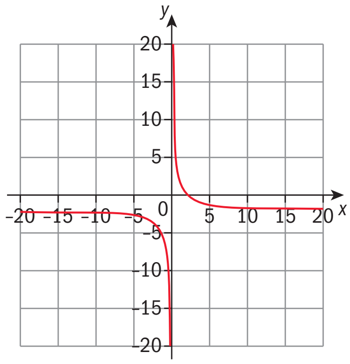
Range $y \in \mathbb{R}$, $y \neq 0$

b



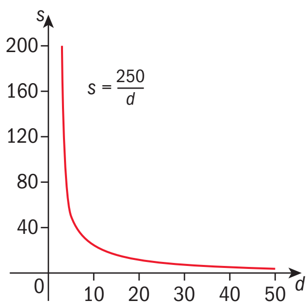
$y = 2$, $x = 0$. Domain $x \in \mathbb{R}$, $x \neq 0$
Range $y \in \mathbb{R}$, $y \neq 2$.

c



$y = -2$, $x = 0$. Domain $x \in \mathbb{R}$, $x \neq 0$.
Range $y \in \mathbb{R}$, $y \neq -2$.

5 **a**

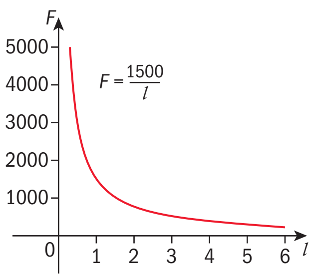


b $10 = \frac{250}{d}$

$d = \frac{250}{10} = 25\text{ m}$

c $s = \frac{250}{100} = 2.5\text{ ms}^{-1}$

6



$F = \frac{1500}{l}$ where l is the length of the lever and the force is measured in newtons.

b $F = \frac{1500}{2} = 750\text{ N}$

c **i** $1000 = \frac{1500}{l}$

$l = \frac{1500}{1000} = 1.5\text{ m}$

ii $2000 = \frac{1500}{l}$

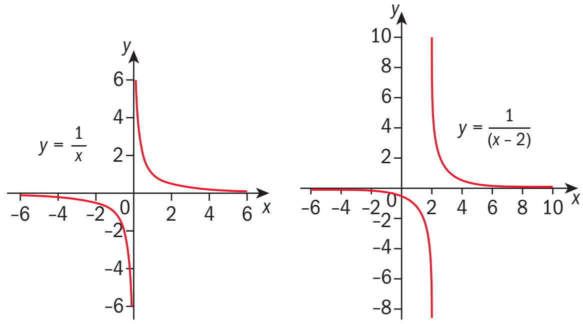
$l = \frac{1500}{2000} = 0.75\text{ m}$

iii $3000 = \frac{1500}{l}$

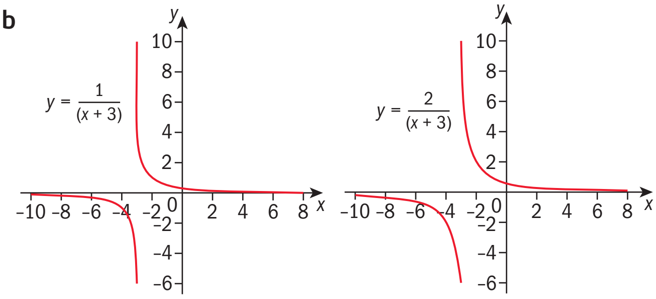
$l = \frac{1500}{3000} = 0.5\text{ m}$

Investigation - graphing rational functions 1

a



b



Rational function	Vertical Asymptote	Horizontal Asymptote	Domain	Range
$y = \frac{1}{x}$	$x = 0$	$y = 0$	$x \in \mathbb{R}$ $x \neq 0$	$y \in \mathbb{R}$, $y \neq 0$
$y = \frac{1}{x-2}$	$x = 2$	$y = 0$	$x \in \mathbb{R}$, $x \neq 2$	$y \in \mathbb{R}$, $y \neq 0$
$y = \frac{1}{x+3}$	$x = -3$	$y = 0$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 0$
$y = \frac{2}{x+3}$	$x = -3$	$y = 0$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 0$

c The vertical asymptote is the solution to the denominator equals zero.

d They are all $y = 0$.

e The domain is $x \in \mathbb{R}$ but x cannot equal the value of the vertical asymptote

f The range is $y \in \mathbb{R}$ but y cannot equal the value of the horizontal asymptote.

Exercise 5C

1 **a** The numerator cannot be zero therefore the horizontal asymptote is $y = 0$.
The denominator is zero when $x = -1$.
Domain $x \in \mathbb{R}, x \neq -1$ Range $y \in \mathbb{R}, y \neq 0$.

b The numerator cannot be zero therefore the horizontal asymptote is $y = 0$.
The denominator is zero when $x = 4$.
Domain $x \in \mathbb{R}, x \neq 4$. Range $y \in \mathbb{R}, y \neq 0$.

c The numerator cannot be zero therefore the horizontal asymptote is $y = 0$.
The denominator is zero when $x = 5$.
Domain $x \in \mathbb{R}, x \neq 5$. Range $y \in \mathbb{R}, y \neq 0$.

d The numerator cannot be zero therefore the horizontal asymptote is $y = 0$.
The denominator is zero when $x = -1$.
Domain $x \in \mathbb{R}, x \neq -1$. Range $y \in \mathbb{R}, y \neq 0$.

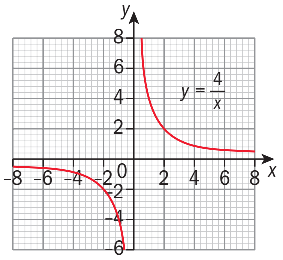
e The numerator cannot be zero therefore the horizontal asymptote is $y = 0$ with a vertical translation of 2 units up. The denominator is zero when $x = -1$. Domain $x \in \mathbb{R}, x \neq -1$. Range $y \in \mathbb{R}, y \neq 2$.

f The numerator cannot be zero therefore the horizontal asymptote is $y = 0$ with a vertical translation of 2 units down. The denominator is zero when $x = -1$. Domain $x \in \mathbb{R}, x \neq -1$. Range $y \in \mathbb{R}, y \neq -2$.

g The numerator cannot be zero therefore the horizontal asymptote is $y = 0$ with a vertical translation of 2 units up. The denominator is zero when $x = 3$. Domain $x \in \mathbb{R}, x \neq 3$. Range $y \in \mathbb{R}, y \neq 2$.

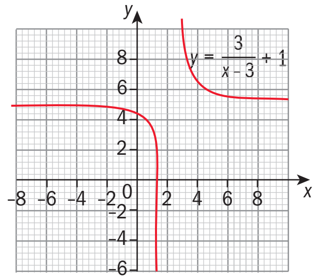
h The numerator cannot be zero therefore the horizontal asymptote is $y = 0$ with a vertical translation of 2 units down. The denominator is zero when $x = -3$. Domain $x \in \mathbb{R}, x \neq -3$. Range $y \in \mathbb{R}, y \neq -2$.

2 a



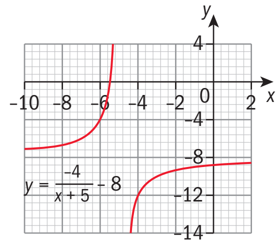
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq 0$.

b



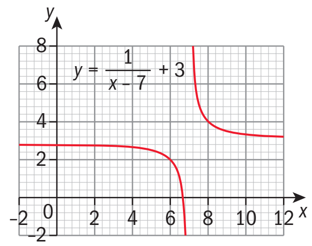
Domain $x \in \mathbb{R}, x \neq 3$. Range $y \in \mathbb{R}, y \neq 1$.

c



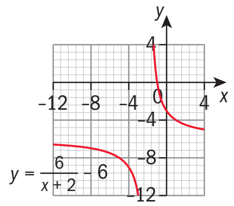
Domain $x \in \mathbb{R}, x \neq -5$. Range $y \in \mathbb{R}, y \neq -8$.

d



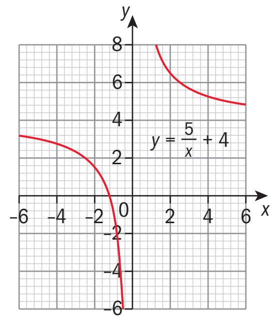
Domain $x \in \mathbb{R}, x \neq 7$. Range $y \in \mathbb{R}, y \neq 3$.

e



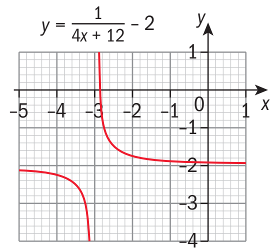
Domain $x \in \mathbb{R}, x \neq -2$. Range $y \in \mathbb{R}, y \neq -6$.

f



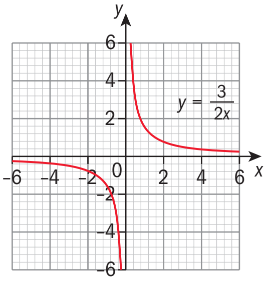
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq 4$.

g



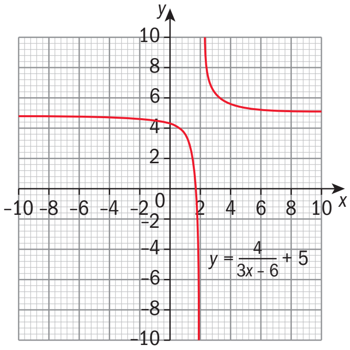
Denominator is zero when $4x + 12 = 0 \Rightarrow x = -3$
Domain $x \in \mathbb{R}, x \neq -3$. Range $y \in \mathbb{R}, y \neq -2$.

h



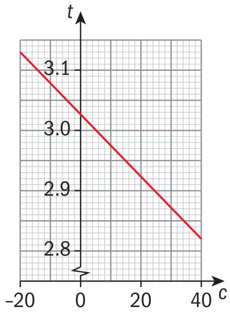
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq 0$.

i



Domain $x \in \mathbb{R}, x \neq 2$. Range $y \in \mathbb{R}, y \neq 5$.

3



Note: at this magnification, the graph might appear to be linear, but you should be aware that this is a reciprocal graph.

$$3 = \frac{1000}{0.6c + 331}$$

$$3(0.6c + 331) = 1000$$

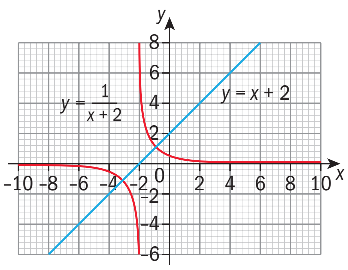
$$1.8c + 993 = 1000$$

$$1.8c = 7$$

$$c = \frac{7}{1.8}$$

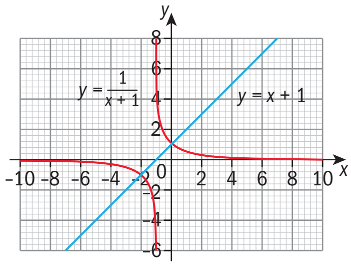
$$c = 3.9°\text{C}$$

4 **a**



The linear function is a line of symmetry for the reciprocal function. The linear function crosses the x-axis at the same place as the vertical asymptote of the reciprocal function.

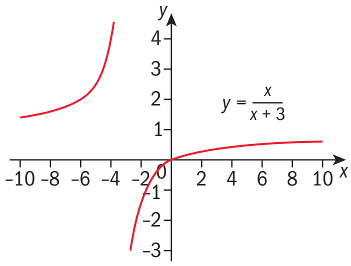
b

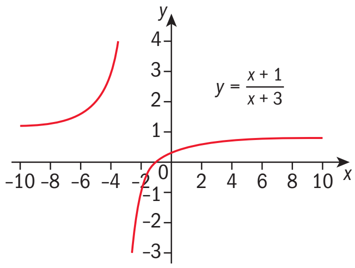


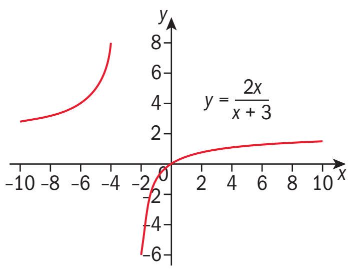
The linear function is a line of symmetry for the reciprocal function. The linear function crosses the x-axis at the same place as the vertical asymptote of the reciprocal function.

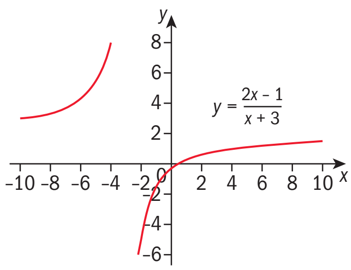
Investigation – graphing rational functions 2

a









b

Rational function	Vertical Asymptote	Horizontal Asymptote	Domain	Range
$y = \frac{x}{x+3}$	$x = -3$	$y = 1$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 1$
$y = \frac{x+1}{x+3}$	$x = -3$	$y = 1$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 1$
$y = \frac{2x}{x+3}$	$x = -3$	$y = 2$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 2$
$y = \frac{2x-1}{x+3}$	$x = -3$	$y = 2$	$x \in \mathbb{R}$, $x \neq -3$	$y \in \mathbb{R}$, $y \neq 2$

c The horizontal asymptotes are the quotient of the x coefficients.

d The vertical asymptote is the solution to the denominator equalling zero. They are all $x = -3$ as the denominators are all $x + 3$. The domain excludes the x-value of the vertical asymptote.

Exercise 5D

1 a Horizontal asymptote when $y = \frac{1}{1} = 1$,

Vertical asymptote when $x - 3 = 0$, $x = 3$.

Domain $x \in \mathbb{R}$, $x \neq 3$. Range $y \in \mathbb{R}, y \neq 1$.

b Horizontal asymptote when $y = \frac{2}{3}$,

Vertical asymptote when $3x - 1 = 0$, $x = \frac{1}{3}$.

Domain $x \in \mathbb{R}, x \neq \frac{1}{3}$. Range $y \in \mathbb{R}, y \neq \frac{2}{3}$.

c Horizontal asymptote when $y = \frac{-3}{-4} = \frac{3}{4}$,

Vertical asymptote when $-4x - 5 = 0$, $x = -\frac{5}{4}$.

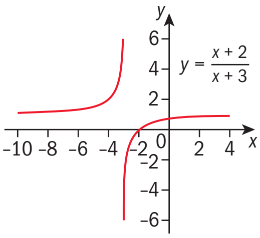
Domain $x \in \mathbb{R}$, $x \neq -\frac{5}{4}$. Range $y \in \mathbb{R}, y \neq \frac{3}{4}$.

d Horizontal asymptote when $y = \frac{34}{16} = \frac{17}{8}$,

Vertical asymptote when $16x + 4 = 0$. $x = -\frac{1}{4}$.

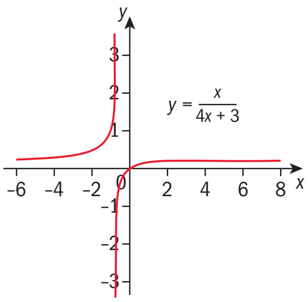
Domain $x \in \mathbb{R}$, $x \neq -\frac{1}{4}$. Range $y \in \mathbb{R}, y \neq \frac{17}{8}$.

2 a



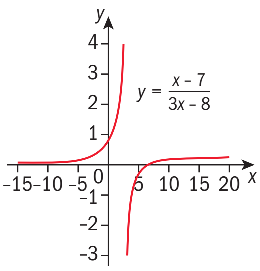
Domain $x \in \mathbb{R}, x \neq -3$ Range $y \in \mathbb{R}, y \neq 1$.

b



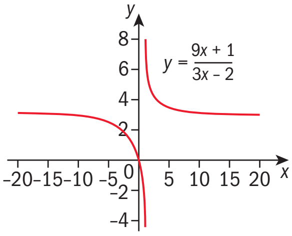
Domain $x \in \mathbb{R}, x \neq -\frac{3}{4}$. Range $y \in \mathbb{R}, y \neq \frac{1}{4}$.

c



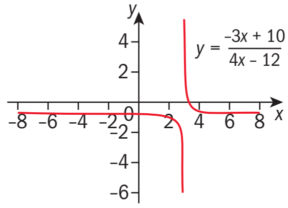
Domain $x \in \mathbb{R}, x \neq \frac{8}{3}$. Range $y \in \mathbb{R}, y \neq \frac{1}{3}$.

d



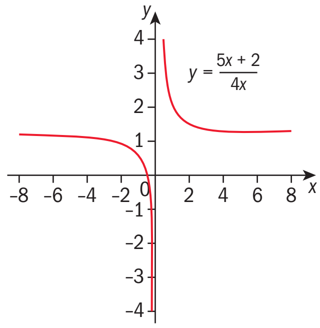
Domain $x \in \mathbb{R}, x \neq \frac{2}{3}$. Range $y \in \mathbb{R}, y \neq 3$.

e



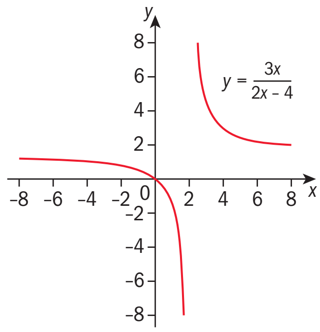
Domain $x \in \mathbb{R}, x \neq 3$. Range $y \in \mathbb{R}, y \neq -\frac{3}{4}$.

f



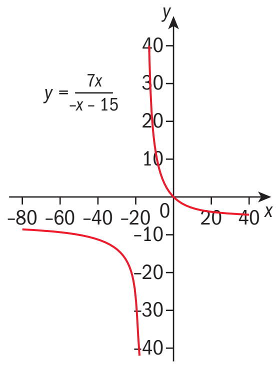
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq \frac{5}{4}$.

g



Domain $x \in \mathbb{R}, x \neq 2$. Range $y \in \mathbb{R}, y \neq \frac{3}{2}$.

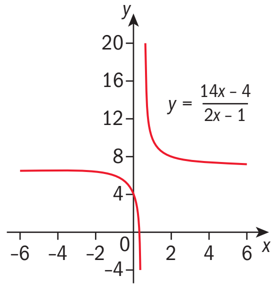
h



Domain $x \in \mathbb{R}, x \neq -15$.

Range $y \in \mathbb{R}, y \neq -7$.

i



Domain $x \in \mathbb{R}, x \neq \frac{1}{2}$. Range $y \in \mathbb{R}, y \neq 7$.

3 a iii Horizontal asymptote at $y = 0$ and vertical asymptote at $x = 0$.

b i Horizontal asymptote at $y = 1$ and vertical asymptote at $x = 2$.

c iv Horizontal asymptote at $y = 1$ and vertical asymptote at $x = 3$.

d ii Horizontal asymptote $y = 0$.
Vertical asymptote at $x = 4$.

4 We need vertical asymptote at $x = -4$
so denominator must be equivalent to $x + 4$
Take $y = \frac{1}{x+4} + a$ for some $a \in \mathbb{R}$.
The numerator cannot be zero. Therefore we need to translate the graph 3 units up to give a horizontal asymptote of $y = 3$. Take $a = 3$.
$y = \frac{1}{x+4} + 3$.

5 a Cost = Set up cost +\$5.50 per T-shirt
$C(x) = 450 + 5.5x$

b Average cost is the total cost divided by the number of T-shirts sold.

$$A(x) = \frac{450 + 5.5x}{x}$$

c Domain is $x > 0$. Since x represents the number of T-shirts produced, only non-negative values make sense. We have to exclude $x = 0$ since $A(x)$ is undefined for $x = 0$ and at $x = 0$ no T-shirts are made.

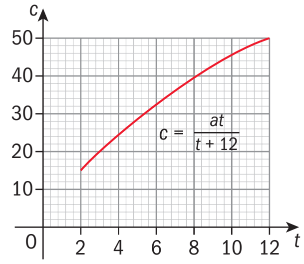
d $x = 0$

e The horizontal asymptote is $y = 5.5$. As the number of shirts produced increases, the set up costs are negligible as a larger number of T-shirts are produced.

6 a

t	2	3	4	5	6	7	8	9	10	11	12
c	$\frac{100}{7}$	20	25	$\frac{500}{17}$	$\frac{100}{3}$	$\frac{700}{19}$	40	$\frac{300}{7}$	$\frac{500}{11}$	$\frac{1100}{23}$	50

b



Label the x axis t and the y axis c.

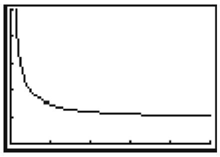
c Approximately 38.5 mg **d** $c = 100$

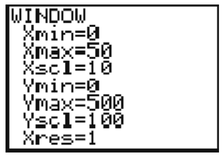
e The children's dose will not exceed 100 mg.

7 a $\frac{550 + 92 \times 15}{15} = \$128.67$

b $C(n) = \frac{550 + 92n}{n}$, where n = number of years and C represents the annual cost.

c The graph of the function is shown below with the window used.

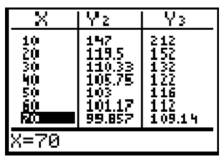


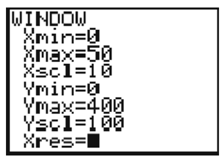


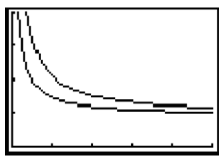
d The asymptotes of the rational function are $n = 0$ and $c = 92$. The n-value asymptote can be seen from the domain; you cannot substitute $n = 0$ into the function. The horizontal asymptote $y = \frac{92}{1} = 92$. From a practical view, the cost of the refrigerator goes to zero over many years; however, the yearly expense of electricity continues.

e The yearly expense of electricity continues no matter how many years the refrigerator works. The cost will never go below the \$92, but the cost approaches \$92 after many years have passed.

f Graphing the two functions $C(n) = \frac{550 + 92n}{n}$ and $C_2(n) = \frac{1200 + 92n}{n}$ together or reviewing a table of values will show the more expensive refrigerator remains more expensive annually although both approach \$92 as n approaches infinity.







Review exercise

1 **i** **a** Vertical asymptote at $x = -2$, horizontal asymptote at $y = 0$.

ii **d** Vertical asymptote at $x = 3$, horizontal asymptote at $y = 0$.

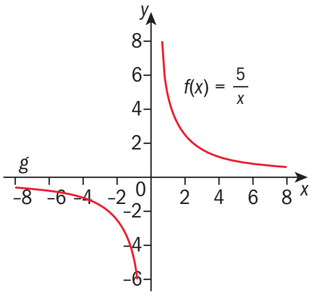
iii **c** Vertical asymptote at $x = 0$, horizontal asymptote at $y = 4$.

iv **e** Vertical asymptote at $x = 0$, horizontal asymptote at $y = -1$.

v **b** Vertical asymptote at $x = 4$, horizontal asymptote at $y = 1$.

vi **f** Vertical asymptote at $x = -4$, horizontal asymptote at $y = 1$.

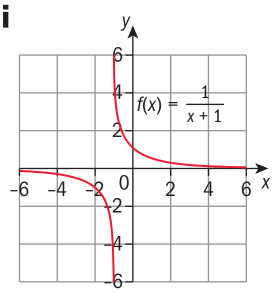
2 **a** **i**



ii $x = 0, y = 0$.

iii Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq 0$.

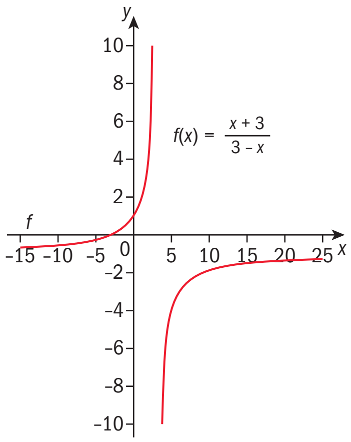
b **i**



ii $x = -1, y = 0$.

iii Domain $x \in \mathbb{R}, x \neq -1$. Range $y \in \mathbb{R}, y \neq 0$.

c **i**



ii $x = 3, y = -1$.

iii Domain $x \in \mathbb{R}, x \neq 3$. Range $y \in \mathbb{R}, y \neq -1$.

3 **a** $x = -4, y = 0$
Domain $x \in \mathbb{R}, x \neq -4$. Range $y \in \mathbb{R}, y \neq 0$.

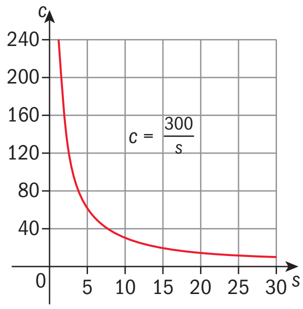
b $x = 0, y = -3$.
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq -3$.

c $x = -6, y = -2$.
Domain $x \in \mathbb{R}, x \neq -6$. Range $y \in \mathbb{R}, y \neq -2$.

d $x = 1, y = 5$.
Domain $x \in \mathbb{R}, x \neq 1$. Range $y \in \mathbb{R}, y \neq 5$.

4 **a** $c = \frac{300}{s}$

b



c The domain and range are limited to real, positive numbers. The domain must also be only integers since this represents the number of students.

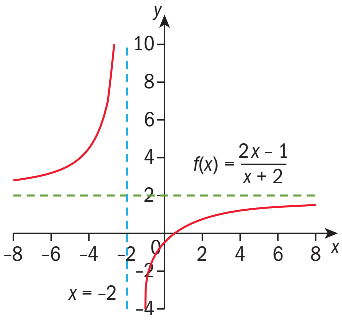
5 **a** **i** $y = 2$

ii $x + 2 = 0, x = -2$.

iii $(-2, 2)$

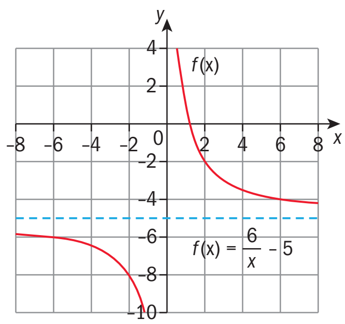
b The y-intercept occurs when $x = 0$. $y = \frac{2(0) - 1}{(0) + 2}$,
$y = -\frac{1}{2}$, So intercept at $\left(0, \frac{-1}{2}\right)$.
The x - intercept occurs when $y = 0$. $\frac{2x - 1}{x + 2} = 0$,
$2x - 1 = 0$, $x = \frac{1}{2}$, So intercept at $\left(\frac{1}{2}, 0\right)$.

c



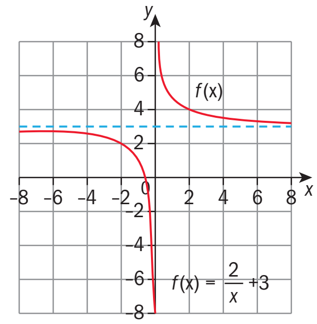
Review exercise

1 **a**



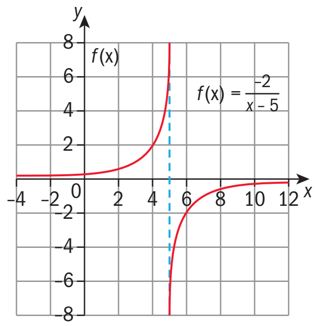
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq -5$.

b



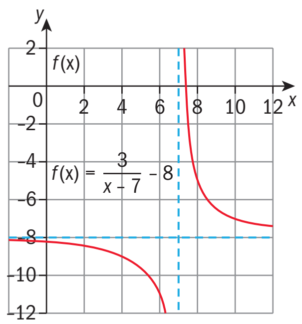
Domain $x \in \mathbb{R}, x \neq 0$. Range $y \in \mathbb{R}, y \neq 3$

c



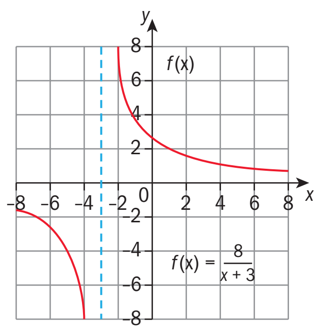
Domain $x \in \mathbb{R}, x \neq 5$. Range $y \in \mathbb{R}, y \neq 0$.

d



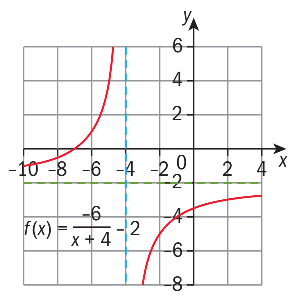
Domain $x \in \mathbb{R}, x \neq 7$. Range $y \in \mathbb{R}, y \neq -8$.

e



Domain $x \in \mathbb{R}, x \neq -3$. Range $y \in \mathbb{R}, y \neq 0$.

f

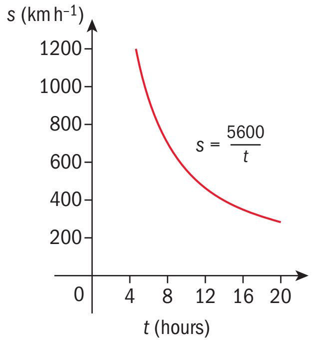


Domain $x \in \mathbb{R}, x \neq -4$. Range $y \in \mathbb{R}, y \neq -2$.

2 a Using the equation $\text{Speed} = \dfrac{\text{distance}}{\text{time}}$,

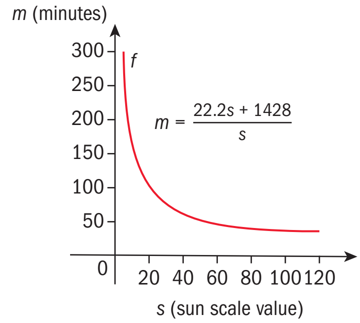
distance = 5600, $s = \dfrac{5600}{t}$

b



c $s = \dfrac{5600}{10} = 560\,\text{km}\,\text{h}^{-1}$

3 a



b i $m = \dfrac{22.2(10) + 1428}{10} = 165$ min

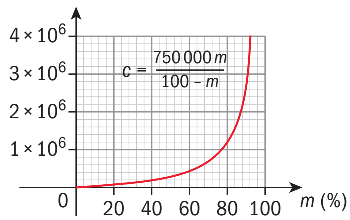
ii $m = \dfrac{22.2(40) + 1428}{40} = 57.9$ min

iii $m = \dfrac{22.2(100) + 1428}{100} = 36.5$ min

c $m = 22.2$

d This represents the number of minutes that can be spent in direct sunlight without skin damage.

4 a



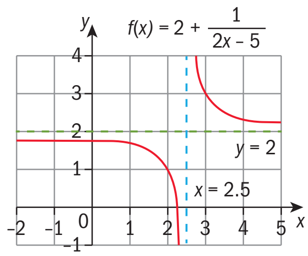
b i $c = \dfrac{750\,000(50)}{100 - 20} = 185\,700$ Thai Baht

ii $c = \dfrac{750\,000(50)}{100 - 50} = 750\,000$ Thai Baht

iii $c = \dfrac{750\,000(90)}{100 - 90} = 6\,750\,000$ Thai Baht.

c No. When $m = 100$ the function is undefined.

5 a



b **i** Vertical asymptote when $2x - 5 = 0$
$\Rightarrow x = \frac{5}{2}$.
Horizontal asymptote at $y = 0$, shifted 2 units up, to give $y = 2$.

ii x-intercept when $y = 0 \Rightarrow \frac{1}{2x-5} = -2$

$$1 = -4x + 10$$
$$4x = 9$$
$$x = \frac{9}{4} = 2.25.$$

x-intercept (2.25, 0).

iii y-intercept when $x = 0 \Rightarrow y = 2 + \frac{1}{-5}$

$$= \frac{9}{5} = 1.8$$

y-intercept (0, 1.8).

These can also be seen from the graph.

6 Patterns, sequences and series

Answers

Skills check

1 **a** $3x - 5 = 5x + 7$
$-2x = 12$
$x = -6$

b $p(2 - p) = -15$
$2p - p^2 = -15$
$p^2 - 2p - 15 = 0$
$(p + 3)(p - 5) = 0$
$p + 3 = 0, p - 5 = 0$
$p = -3, p = 5$

c $2^n + 9 = 41$
$2^n = 32 = 2^5$
$n = 5$

2 **a** $6m + 8k = 30$
$8k = 30 - 6m$
$k = \frac{30 - 6m}{8}$
$k = \frac{15 - 3m}{4}$

b $2pk - 5 = 3$
$2pk = 8$
$k = \frac{8}{2p}$
$k = \frac{4}{p}$

3 **a** $T = 2x(x + 3y)$
$T = 2(3)(3 + 3(5))$
$T = 6(18)$
$T = 108$

b $T = 2\,(4.7)(4.7 + 3(-2))$
$T = 9.4\,(-1.3)$
$T = -\,12.22$

4 **a** $m = 2^x - y^3$
$m = 2^5 - 3^3$
$m = 32 - 27$
$m = 5$

b $m = 2^3 - (-\,2)^3$
$m = 8 - (-\,8)$
$m = 16$

c $m = 2^{-5} - \left(\frac{1}{2}\right)^3$
$m = \frac{1}{32} - \frac{1}{8}$
$m = -\frac{3}{32}$

Investigation – saving money

Week Number	Weekly Savings	Total Savings
1	20	20
2	25	45
3	30	75
4	35	110
5	40	150
6	45	195
7	50	245
8	55	300

Joel saves \$65 in the 10th week, \$100 in the 17th week
He will save \$7670 in the first year.
It will take 17 weeks to save over \$1000.
$M = 20 + 5\,(n - 1)$ OR $M = 15 + 5n$
$T = \frac{5n^2 + 35n}{2}$ OR $T = \frac{n(35 + 5n)}{2}$ OR $T = \frac{5n(7 + n)}{2}$

Exercise 6A

1 **a** 19, 23, 27 *(add 4 to the previous term)*

b 16, 32, 64 *(multiply previous term by 2)*

c 18, 24, 31 *(add 1, add 2, add 3, add 4, and so on...)*

d 80, −160, 320 *(multiply previous term by −2)*

e $\frac{9}{14}, \frac{11}{17}, \frac{13}{20}$ *(numerator increases by 2, denominator increases by 3)*

f 6.01234, 6.012345, 6.0123456 (The decimal places are consecutive integers).

2 **a** $u_1 = 10, u_2 = 3(10) = 30, u_3 = 3(30) = 90,$
$u_4 = 3(90) = 270$

b $u_1 = 3, u_2 = 2(3) + 1 = 7, u_3 = 2(7) + 1 = 15,$
$u_4 = 2(15) + 1 = 31$

c $u_1 = \frac{3}{4}, u_2 = \frac{2}{3}\left(\frac{3}{4}\right) = \frac{1}{2}, u_3 = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3},$
$u_4 = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{2}{9}$

d $u_1 = x, u_2 = (x)^2 = x^2, u_3 = (x^2)^2 = x^4,$
$u_4 = (x^4)^2 = x^8$

3 **a** $u_1 = 2$ and $u_{n+1} = u_n + 2$ (since each term is found by adding 2 to the previous term)

b $u_1 = 1$ and $u_{n+1} = 3u_n$ (since each term is found by multiplying the previous term by 3).

c $u_1 = 64$ and $u_{n+1} = \frac{u_n}{2}$ (since each term is found by multiplying the previous term by $\frac{1}{2}$).

d $u_1 = 7$ and $u_{n+1} = u_n + 5$ (since each term is found by adding 5 to the previous term).

4 **a** $u_n = 3^n$. $u_1 = 3^1 = 3$, $u_2 = 3^2 = 9$, $u_3 = 3^3 = 27$, $u_4 = 3^4 = 81$

b $u_n = -6n + 3$. $u_1 = -6(1) + 3 = -3$, $u_2 = -6(2) + 3 = -9$, $u_3 = -6(3) + 3 = -15$ $u_4 = -6(4) + 3 = -21$

c $u_n = 2^{n-1}$. $u_1 = 2^0 = 1$, $u_2 = 2^1 = 2$, $u_3 = 2^2 = 4$, $u_4 = 2^3 = 8$.

d $u_n = n^n$. $u_1 = 1^1 = 1$, $u_2 = 2^2 = 4$, $u_3 = 3^3 = 27$, $u_4 = 4^4 = 256$.

5 **a**
$$\begin{array}{r|llllll} \textit{Term number} & 1 & 2 & 3 & 4 & \dots & n \\ & \downarrow\times 2 & \downarrow\times 2 & \downarrow\times 2 & \downarrow\times 2 & & \downarrow\times 2 \\ \textit{Term} & 2 & 4 & 6 & 8 & \dots & 2n \end{array}$$

To get each term, we multiply the term number by 2. So $u_n = 2n$

b
$$\begin{array}{r|llllll} \textit{Term number} & 1 & 2 & 3 & 4 & \dots & n \\ & \downarrow 3^0 & \downarrow 3^1 & \downarrow 3^2 & \downarrow 3^3 & & \downarrow 3^{n-1} \\ \textit{Term} & 1 & 3 & 9 & 27 & & 3^{n-1} \end{array}$$

To get each term u_n, we raise 3 to the power of $n - 1$. So $u_n = 3^{n-1}$

c
$$\begin{array}{r|llllll} \textit{Term number} & 1 & 2 & 3 & 4 & \dots & n \\ & \downarrow 2^{7-1} & \downarrow 2^{7-2} & \downarrow 2^{7-3} & \downarrow 2^{7-4} & & \downarrow 2^{7-n} \\ \textit{Term} & 64 & 32 & 16 & 8 & & 2^{7-n} \end{array}$$

To get each term, u_n we raise 2 to the power $(7 - n)$. Thus, $u_n = 2^{7-n}$

d
$$\begin{array}{r|lllll} \textit{Term number} & 1 & 2 & 3 & 4 & n \\ & \downarrow(5\times1)+2 & \downarrow(5\times2)+2 & \downarrow(5\times3)+2 & \downarrow(5\times4)+2 & \downarrow(5\times3)+2 \\ \textit{Term} & 7 & 12 & 17 & 22 & 5n+2 \end{array}$$

To get each term, u_n, we multiply n by 5 and add 2.

e
$$\begin{array}{r|lllll} \textit{Term number} & 1 & 2 & 3 & 4 & n \\ & \downarrow\frac{1}{1+1} & \downarrow\frac{2}{2+1} & \downarrow\frac{3}{3+1} & \downarrow\frac{4}{4+1} & \downarrow\frac{n}{n+1} \\ \textit{Term} & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{n}{n+1} \end{array}$$

To get each term, u_n, we divide n by $n + 1$. So $u_n = \frac{n}{n+1}$.

f
$$\begin{array}{r|lllll} \textit{Term number} & 1 & 2 & 3 & 4 & n \\ & \downarrow 1\times x & \downarrow 2\times x & \downarrow 3\times x & \downarrow 4\times x & \downarrow n\times x \\ \textit{Term} & x & 2x & 3x & 4x & \downarrow n\times x \end{array}$$

To get each term, u_n, we multiply x by n. So $u_n = nx$.

6 **a** 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 the 15th term is 610

b $u_1 = 1$, $u_2 = 1$, and $u_{n+1} = u_n + u_{n-1}$

Exercise 6B

1 **a** **i** $u_n = u_1 + (n - 1)d$. $u_1 = 3$ and $d = 3$,
$u_{15} = 3 + (15 - 1)3$
$u_{15} = 3 + (14)3$
$u_{15} = 45$

ii $u_n = 3 + (n - 1)3$
$u_n = 3n$

b **i** $u_n = u_1 + (n - 1)d$. $u_1 = 25$ and $d = 15$,
$u_{15} = 25 + (15 - 1)15$
$u_{15} = 25 + (14)15$
$u_{15} = 235$

ii $u_n = 25 + (n - 1)15$
$u_n = 15n + 10$

c **i** $u_n = u_1 + (n - 1)d$. $u_1 = 36$ and $d = 5$,
$u_{15} = 36 + (15 - 1)5$
$u_{15} = 36 + (14)5$
$u_{15} = 106$

ii $u_n = 36 + (n - 1)5$
$u_n = 5n + 31$

d **i** $u_n = u_1 + (n - 1)d$. $u_1 = 100$ and $d = -13$,
$u_{15} = 100 + (15 - 1)(-13)$
$u_{15} = 100 + (14)(-13)$
$u_{15} = -82$

ii $u_n = 100 + (n - 1)(-13)$
$u_n = 113 - 13n$

e **i** $u_n = u_1 + (n - 1)d$. $u_1 = 5.6$ and $d = 0.6$,
$u_{15} = 5.6 + (15 - 1)(0.6)$
$u_{15} = 5.6 + (14)(0.6)$
$u_{15} = 14$

ii $u_n = 5.6 + (n - 1)(0.6)$
$u_n = 0.6n + 5$

f **i** $u_n = u_1 + (n - 1)d$. $u_1 = x$ and $d = a$,
$u_{15} = x + (15 - 1)(a)$
$u_{15} = x + 14a$

ii $u_n = x + (n - 1)(a)$
$u_n = x + an - a$

2 **a** $u_1 + (n - 1)d = u_n$
$5 + (n - 1)5 = 255$
$5n = 255$
$n = 51$

b $u_1 + (n - 1)d = u_n$
$4.8 + (n - 1)(0.2) = 38.4$
$0.2n + 4.6 = 38.4$
$0.2n = 38$
$n = 169$

c $u_1 + (n - 1)d = u_n$
$\frac{1}{2} + (n-1)\left(\frac{3}{8}\right) = 14$
$\frac{3}{8}n + \frac{1}{8} = 14$
$\frac{3}{8}n = \frac{111}{8}$
$n = 37$

d $u_1 + (n-1)d = u_n$
$250 + (n-1)(-29) = -156$
$-29n + 279 = -156$
$-29n = -435$
$n = 15$

e $u_1 + (n-1)d = u_n$
$2m + (n-1)(3m) = 80m$
$3mn - m = 80m$
$3mn = 81m$
$n = 27$

f $u_1 + (n-1)d = u_n$
$x + (n-1)(2x+3) = 19x + 27$
$n(2x+3) - x - 3 = 19x + 27$
$n(2x+3) = 20x + 30 = 10(2x+3)$
$n = 10$

Exercise 6C

1 $u_{15} = 19 + (15-1)d = 31.6$
$19 + 14d = 31.6$
$14d = 12.6$
$d = 0.9$

2 $u_{10} = u_1 + (10-1)d = 37$
$u_1 + 9d = 37$ *(call this equation #1)*
$u_{21} = u_1 + (21-1)d = 4$
$u_1 + 20d = 4$ *(call this equation #2)*
(solve using simultaneous equation solver on GDC)
$u_1 = 64,\ d = -3$

3 $3 + 2d = 8$
$d = 2.5$
$x = 3 + 2.5$
$x = 5.5$

4 since $u_1 + d = u_2$, $m + d = 13$ *(call this equation #1)*
since $u_2 + d = u_3$, $13 + d = 3m - 6$
$3m - d = 19$ *(call this equation #2)*
(add equations #1 and #2)
$4m = 32$
$m = 8$

Exercise 6D

1 a $r = \frac{u_2}{u_1} = \frac{1}{2}$
$u_n = u_1\,(r)^{n-1}$
$u_7 = 16\left(\frac{1}{2}\right)^6$
$u_7 = 16\left(\frac{1}{64}\right)$
$u_7 = \frac{1}{4}$

b $r = \frac{u_2}{u_1} = -3$
$u_n = u_1\,(r)^{n-1}$
$u_7 = -4(-3)^6$
$u_7 = -4(729)$
$u_7 = -2916$

c $r = \frac{u_2}{u_1} = 10$
$u_n = u_1\,(r)^{n-1}$
$u_7 = 1(10)^6$
$u_7 = 1000000$

d $r = 0.4$
$u_n = u_1\,(r)^{n-1}$
$u_7 = 25(0.4)^6$
$u_7 = 25(0.004096)$
$u_7 = 0.1024$

e $r = 3x$
$u_n = u_1\,(r)^{n-1}$
$u_7 = 2(3x)^6$
$u_7 = 2(729x^6)$
$u_7 = 1458x^6$

f $r = \frac{u_2}{u_1} = \frac{a^6b^2}{a^7b} = \frac{b}{a}$
$u_n = u_1(r)^{n-1}$
$u_7 = a^7b\left(\frac{b}{a}\right)^6$
$u_7 = a^7b\left(\frac{b^6}{a^6}\right)$
$u_7 = ab^7$

Exercise 6E

1 $u_5 = u_2\,(r)^3$
$3.2 = 50r^3$
$r^3 = \frac{3.2}{50} = \frac{8}{125}$
$r = \frac{2}{5} = 0.4$
$u_1(r) = u_2$
$u_1(0.4) = 50$
$u_1 = 125$

2 $u_6 = u_3\,(r)^3$
$144 = -18r^3$
$r^3 = \frac{144}{-18} = -8$
$r = -2$
$u_1\,(r)^2 = u_3$
$u_1(-2)^2 = -18$
$u_1 = \frac{-18}{4} = -4.5$

3 a We want n such that
$u_1\,(r)^{n-1} > 1000$
$u_1 = 16$ and $r = 1.5$, so
$16(1.5)^{n-1} > 1000$
$(1.5)^{n-1} > 62.5$
$n = 12$

this equation can be solved using the table on the GDC, or by using logarithms

b $u_1\,(r)^{n-1} > 1000.$
$u_1 = 1,\ r = 2.4,$
$1(2.4)^{n-1} > 1000$
$(2.4)^{n-1} > 1000$
$n = 9$

c $u_1(r)^{n-1} > 1000$. $u_1 = 112$, $r = \frac{-168}{112}$,

$112\left(-\frac{3}{2}\right)^{n-1} > 1000$

$\left(-\frac{3}{2}\right)^{n-1} > \frac{125}{14}$,

$n = 7$

d $u_1(r)^{n-1} > 1000$. $u_1 = 50$, $r = \frac{55}{50}$, $= 1.1$

$50(1.1)^{n-1} > 1000$

$(1.1)^{n-1} > 20$

$n = 33$

4 We know $u_3 = u_1 r^2$, so $9r^2 = 144$

$r^2 = 16$

$r = \pm 4$

If $r = 4$	If $r = -4$
$u_2 = u_1(r)$	$u_2 = u_1(r)$
$u_2 = 9(4) = 36$	$u_2 = 9(-4) = -36$

5 $18r^2 = 40.5$

$r^2 = \frac{40.5}{18} = 2.25$

$r = \pm 1.5$

If $r = 1.5$	If $r = -1.5$
$p = 18(1.5) = 27$	$p = 18(-1.5) = -27$

6 $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\frac{4x+4}{7x-2} = \frac{3x}{4x+4}$

$(4x + 4)(4x + 4) = (3x)(7x - 2)$

$16x^2 + 32x + 16 = 21x^2 - 6x$

$5x^2 - 38x - 16 = 0$

$x = 8$

This quadratic equation can be solved using the polynomial root finder on your GDC. There are 2 roots, but the question asks for the positive value of x.

Exercise 6F

1 **a** $\sum_{n=1}^{8} n$

b Write the series as $3^2 + 4^2 + 5^2 + 6^2 + 7^2$

Then sum is $\sum_{n=3}^{7} n^2$

c Write the series as

$[29 - 2(1)] + [29 + 2(2)] + [29 - 2(3)]$

$+ [29 - 2(4)] + [29 - 2(5)] + [29 - 2(6)]$

Then sum is $\sum_{n=1}^{6} 29 - 2n$

d Write the series as $240\left(\frac{1}{2}\right)^0 + 240\left(\frac{1}{2}\right)^1 + 240\left(\frac{1}{2}\right)^2$

$+ 240\left(\frac{1}{2}\right)^3 + 240\left(\frac{1}{2}\right)^4 + 240\left(\frac{1}{2}\right)^5$

Then sum is $\sum_{n=1}^{6} 240\left(\frac{1}{2}\right)^{n-1}$

e The sum is $\sum_{n=5}^{10} nx$

f Write the series as $(3(1) + 1) + (3(2) + 1) +$ $(3(3) + 1) + (3(4) + 1) + \cdots + (3(8) + 1)$

The sum is $\sum_{n=1}^{18} 3n + 1$

g Write the series as $3^0 + 3^1 + 3^2 + 3^3 + \cdots + 3$

The sum is $\sum_{n=1}^{11} 3^{n-1}$

h The sum is $\sum_{n=1}^{5} na^n$

2 **a** $\sum_{n=1}^{8}(3n+1) = (3(1) + 1) + (3(2) + 1) + (3(3) + 1)$
$+ (3(4) + 1) + (3(5) + 1) + (3(6) + 1)$
$+ (3(7) + 1) + (3(8) + 1)$
$= 4 + 7 + 10 + 13 + 16 + 19 + 22 + 25$

b $\sum_{a=1}^{5} 4^a = 4^1 + 4^2 + 4^3 + 4^4 + 4^5$
$= 4 + 16 + 64 + 256 + 1024$

c $\sum_{r=3}^{7}(5(2^r)) = 5(8) + 5(16) + 5(32) + 5(64) + 5(128)$
$= 40 + 80 + 160 + 320 + 640$

d $\sum_{n=5}^{a} x^n = x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11}$

3 **a** $\sum_{n=1}^{a}(8n-5) = (8(1) - 5) + (8(2) - 5) + (8(3) - 5)$
$+ (8(4) - 5) + (8(5) - 5) + (8(6) - 5)$
$+ (8(7) - 5) + (8(8) - 5) + (8(9) - 5)$
$= 3 + 11 + 19 + 27 + 35 + 43 + 51$
$+ 59 + 67 = 315$

b $\sum_{r=1}^{5}(3^r) = 3^1 + 3^2 + 3^3 + 3^4 + 3^5$
$= 3 + 9 + 27 + 81 + 243$
$= 363$

c $\sum_{m=1}^{7} m^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$
$= 1 + 4 + 9 + 16 + 25 + 36 + 49$
$= 140$

d $\sum_{x=4}^{10} 7x - 4 = (7(4) - 4) + (7(5) - 4) + (7(6) - 4)$
$+ (7(7) - 4) + (7(8) - 4) + (7(9) - 4)$
$+ (7(10) - 4)$
$= 24 + 31 + 38 + 45 + 52 + 59 + 66$
$= 315$

Exercise 6G

1 $S_{12} = \frac{12}{2}\left(2(3) + (12 - 1)(3)\right)$

$S_{12} = 6(6 + 33) = 234$

2 $S_{18} = \frac{18}{2}\left(2(2.6) + (18 - 1)(0.4)\right)$

$S_{18} = 9(5.2 + 6.8) = 108$

3 $S_{27} = \frac{27}{2}\left(2(100) + (27 - 1)(-6)\right)$

$S_{27} = 13.5(200 - 156) = 594$

4 $S_{16} = \frac{16}{2}\left(2(2 - 5x) + (16 - 1)(1 + x)\right)$

$S_{16} = 8(4 - 10x + 15 + 15x)$

$S_{16} = 8(19 + 5x) = 40x + 152$

5 $u_1 = 120$, $d = -4$. we know $u_1 + (n-1)d = u_n$, so
$120 + (n-1)(-4) = 28$
$124 - 4n = 28$
$4n = 96$
$n = 24$
$S_{24} = \frac{24}{2}(120 + 28)$
$S_{24} = 12(148) = 1776$

6 $u_1 = 15$ and $d = 7$. From $u_n = u_1 + (n-1)d$,
$15 + (n-1)(7) = 176$
$7n + 8 = 176$
$7n = 168$
$n = 24$
$S_{24} = \frac{24}{2}(15 + 176)$
$S_{24} = 12(191) = 2292$

Exercise 6H

1 We know $S_n = \frac{n}{2}(2u_1 + (n-1)d)$. Here, $u_1 = 4$ so
$S_{30} = \frac{30}{2}(2(4) + 29d) = 1425$
$15(8 + 29d) = 1425$
$120 + 435d = 1425$
$435d = 1305$
$d = 3$

2 a Using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ with $u_1 = 1$ and $d = 6$,
$S_n = \frac{n}{2}(2(1) + (n-1)(6))$
$S_n = \frac{n}{2}(6n - 4)$
$3n^2 - 2n$

b $3n^2 - 2n = 833$
$3n^2 - 2n = 833 = 0$
(use polynomial equation solver on GDC)
$n = 17$

3 a Using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ with $u_1 = -30$ and $d = 3.5$,
$S_n = \frac{n}{2}(2(-30) + (n-1)(3.5))$
$S_n = \frac{n}{2}(3.5n - 63.5)$
$1.75n^2 - 31.75n$

b $1.75n^2 - 31.75n = 105$
$1.75n^2 - 31.75n - 105 = 0$
(use polynomial equation solver on GDC)
$n = 21$

4 a If we write as an arithmetic progression,
$u_1 = 500$, $u_2 = 600$, $u_3 = 700$, ...
December 2012 will be the 12th month. Using $u_n = u_1 + (n-1)d$ with $u_1 = 500$ and $d = 100$, we see that for $n = 12$,
$u_{12} = 500 + (12 - 1)100$
$= 500 + 1100$
$= 1600$

b Using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
$S_{12} = \frac{12}{2}(1000 + (12 - 1)100)$
$= 12600$

5 Second term given by $u_2 = u_1 + d$
5th term given by $u_5 = u_1 + 4d$.
we're told $u_2 = 4u_5$, so
$u_1 + d = 4(u_1 + 4d)$
$3u_1 + 15d = 0.$ (equation 1)
Also, using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$, since we know $S_{10} = -20$
we have $-20 = \frac{10}{2}(2u_1 + 9d)$
$-20 = 10u_1 + 45d.$ (equation 2)
(use simultaneous equation solver on GDC)
$a = -20$, $d = 4$.

6 Since $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ and $S_{12} = 10S_3$,
$\frac{12}{2}(2(5) + 11d) = 10\left(\frac{3}{2}(2(5) + 2d)\right)$
$6(10 + 11d) = 15(10 + 2d)$
$60 + 66d = 150 + 30d$
$36d = 90$
$d = 2.5$
$S_{20} = \frac{20}{2}(2(5) + 19(2.5))$
$S_{20} = 10(10 + 47.5)$
$S_{20} = 575$

Exercise 6I

1 a $u_1 = 0.5$ and $r = 3$. Using $S_n = \frac{u_1(r^n - 1)}{r - 1}$,
$S_{12} = \frac{0.5(3^{12} - 1)}{3 - 1}$
$S_{12} = \frac{265720}{2} = 132\,860$

b $S_{12} = \frac{0.3(2^{12} - 1)}{2 - 1}$
$S_{12} = \frac{0.3(4095)}{1} = 1228.5$

c $S_{12} = \frac{64(1 - (-0.5)^{12})}{1 - (-0.5)} = 42.656\,25$

d $S_{12} = \frac{(x+1)(2^{12} - 1)}{2 - 1}$
$S_{12} = \frac{(x+1)(4095)}{1} = 4095x + 4095$

2 a $S_{20} = \frac{0.25(3^{20} - 1)}{3 - 1}$
$S_{20} = \frac{0.25(3486784400)}{2} = 435\,848\,050$

b $S_{20} = \frac{\frac{16}{9}\left(\left(\frac{3}{2}\right)^{20} - 1\right)}{\frac{3}{2} - 1} \approx 11819.58$

c $S_{20} = \frac{3(1 - (-2)^{20})}{1 - (-2)}$
$S_{20} = \frac{3(-1048575)}{3} = -1048\,575$

d $S_{20} = \dfrac{(\log a)\left((2)^{20}-1\right)}{2-1}$

$S_{20} = \dfrac{1048575(\log a)}{1} = \log\left(a^{1\,048\,575}\right)$

3 a i $1024\left(\dfrac{3}{2}\right)^{n-1} = 26\,244$

This type of equation can be solved using logarithms.

$\left(\dfrac{3}{2}\right)^{n-1} = \dfrac{6561}{256}$

$n - 1 = 8$

$n = 9$

ii $S_9 = \dfrac{1024\left(\left(\dfrac{3}{2}\right)^9 - 1\right)}{\dfrac{3}{2}-1} = 76\,684$

b i $2.7(4)^{n-1} = 2764.8$

$(4)^{n-1} = 1024$

$n - 1 = 5$

$n = 6$

ii $S_6 = \dfrac{2.7\left((4)^6-1\right)}{4-1} = \dfrac{2.7(4095)}{3} = 3685.5$

c i $\dfrac{125}{128}\left(\dfrac{2}{5}\right)^{n-1} = \dfrac{1}{625}$

$\left(\dfrac{2}{5}\right)^{n-1} = \dfrac{128}{78125}$

$n - 1 = 7$

$n = 8$

ii $S_8 = \dfrac{\dfrac{125}{128}\left(1-\left(\dfrac{2}{5}\right)^8\right)}{1-\dfrac{2}{5}} = 1.626\,5375$

d i $590.49\left(\dfrac{1}{3}\right)^{n-1} = 0.01$

$\left(\dfrac{1}{3}\right)^{n-1} = \dfrac{0.01}{590.49}$

$n - 1 = 10$

$n = 11$

ii $S_{11} = \dfrac{590.49\left(1-\left(\dfrac{1}{3}\right)^{11}\right)}{1-\dfrac{1}{3}} = 885.73$

Exercise 6J

1 a $u_1 = 25.6\ r = 1.5$

$S_n = \dfrac{25.6\left((1.5)^n - 1\right)}{1.5-1} = \dfrac{25.6\left((1.5)^n-1\right)}{0.5}$

$= 51.2\left((1.5)^n - 1\right) > 400$

$S_5 = 337.6,\ S_6 = 532$

$n = 6$

You can get these values from the tables on your GDC.

b $S_n = \dfrac{14\left(1-(-3)^n\right)}{1-(-3)} = \dfrac{14\left(1-(-3)^n\right)}{4} = 3.5\left(1-(-3)^n\right) > 400$

$S_4 = -280,\ S_5 = 854$

$n = 5$

c $S_n = \dfrac{\dfrac{2}{3}\left(\left(\dfrac{4}{3}\right)^n - 1\right)}{\dfrac{4}{3}-1} = \dfrac{\dfrac{2}{3}\left(\left(\dfrac{4}{3}\right)^n-1\right)}{\dfrac{1}{3}} = 2\left(\left(\dfrac{4}{3}\right)^n - 1\right) > 400$

$S_{18} = 353.75\ldots,\ S_{19} = 471.005\ldots$

$n = 19$

d $S_n = \dfrac{0.02\left((10)^n-1\right)}{10-1} = \dfrac{0.02\left((10)^n-1\right)}{9} > 400$

$S_5 = 222.22,\ S_6 = 2222.22$

$n = 6$

2 $u_8 = u_3 r^{(8-3)} = u_3 r^5.$

$1.2r^5 = 291.6$

$r^5 = 243$

$r = 3$

In order to find S_{10}, we must first find u_1. Now $u_1 r^2 = u_3$, so $u_1(3^2) = 1.2$

$u_1 = \dfrac{2}{15}$ So substituting $u_1 = \dfrac{2}{15}$ and $r = 3$ into

$S_{10} = \dfrac{u_1(r^{10}-1)}{r-1}$, we have

$S_{10} = \dfrac{\dfrac{2}{15}\left((3)^{10}-1\right)}{3-1} = \dfrac{\dfrac{2}{15}\left((3)^{10}-1\right)}{2} = \dfrac{59048}{15}$

3 $S_4 = \dfrac{u_1\left(r^4-1\right)}{r-1} = 20 \rightarrow \dfrac{\left(r^4-1\right)}{20} = \dfrac{r-1}{u_1}$

$S_7 = \dfrac{u_1\left(r^7-1\right)}{r-1} = 546.5 \rightarrow \dfrac{\left(r^7-1\right)}{546.5} = \dfrac{r-1}{u_1}$

$\dfrac{r^4-1}{20} = \dfrac{r^7-1}{546.5} \rightarrow 20\left(r^7-1\right) = 546.5\left(r^4-1\right)$

$20r^7 - 546.5r^4 + 526.5 = 0$

$r = 3$

4 a $r = \dfrac{\left(\dfrac{1}{8}\right)}{\left(\dfrac{1}{12}\right)} = \dfrac{12}{8} = \dfrac{3}{2}$

b $S_n = \dfrac{\dfrac{1}{12}\left(\left(\dfrac{3}{2}\right)^n-1\right)}{\dfrac{3}{2}-1} = \dfrac{\dfrac{1}{12}\left(\left(\dfrac{3}{2}\right)^n-1\right)}{\dfrac{1}{2}} = \dfrac{1}{6}\left(\left(\dfrac{3}{2}\right)^n-1\right) > 800$

$S_{20} = 554.04\ldots,\ S_{21} = 831.147\ldots$

$n = 21$

5 $S_3 = \dfrac{u_1\left(r^3-1\right)}{r-1} = 304 \rightarrow \dfrac{\left(r^3-1\right)}{304} = \dfrac{r-1}{u_1}$

$S_6 = \dfrac{u_1\left(r^6-1\right)}{r-1} = 1330 \rightarrow \dfrac{\left(r^6-1\right)}{1330} = \dfrac{r-1}{u_1}$

$\dfrac{\left(r^3-1\right)}{304} = \dfrac{\left(r^6-1\right)}{1330} \rightarrow 304\left(r^6-1\right) = 1330\left(r^3-1\right)$

$304r^6 - 1330r^3 + 1026 = 0$

$r = 1.5$

$\dfrac{u_1\left((1.5)^3-1\right)}{1.5-1} = 304 \rightarrow u_1(2.375) = 152$

$u_1 = 64$

Now using $S_7 = \dfrac{u_1(r^7-1)}{r-1}$ with $u_1 = 64,\ r = 1.5$

$S_7 = \dfrac{64\left((1.5)^7-1\right)}{1.5-1} = 128\left((1.5)^7-1\right)$

$S_7 = 2059$

6 $S_4 = \frac{u_1(r^4-1)}{r-1}$ and $S_2 = \frac{u_1(r^2-1)}{r-1}$. since $S_4 = 10S_2$,

$\frac{u_1(r^4-1)}{r-1} = 10\left(\frac{u_1(r^2-1)}{r-1}\right)$ multiply both side by $\frac{r-1}{u_1}$

$r^4 - 1 = 10r^2 - 10$

$r^4 - 10r^2 + 9 = 0$

$r = 3$

Exercise 6K

1 $|r| < 1$ means that a germetiric series will be convergent.

2 **a** $S_4 = \frac{144\left(1-\left(\frac{1}{3}\right)^4\right)}{\left(1-\frac{1}{3}\right)} = 213.\overline{3}$

$S_7 = \frac{144\left(1-\left(\frac{1}{3}\right)^7\right)}{\left(1-\frac{1}{3}\right)} \approx 215.9$

$S_\infty = \frac{144}{\left(1-\frac{1}{3}\right)} = 216$

b $S_4 = \frac{500\left(1-(0.8)^4\right)}{(1-0.8)} = 1476$

$S_7 = \frac{500\left(1-0.8^7\right)}{(1-0.8)} = 1975.712$

$S_\infty = \frac{500}{(1-0.8)} = 2500$

c $S_4 = \frac{80\left(1-0.1^4\right)}{(1-0.1)} = 88.88$

$S_7 = \frac{80\left(1-(0.1)^7\right)}{(1-0.1)} = 88.8888$

$S_\infty = \frac{80}{(1-0.1)} = 88.\overline{8}$

d $S_4 = \frac{\frac{9}{2}\left(1-\left(\frac{2}{3}\right)^4\right)}{\left(1-\frac{2}{3}\right)} = 10.8\overline{3}$

$S_7 = \frac{\frac{9}{2}\left(1-\left(\frac{2}{3}\right)^7\right)}{\left(1-\frac{2}{3}\right)} \approx 12.71$

$S_\infty = \frac{\left(\frac{9}{2}\right)}{\left(1-\frac{2}{3}\right)} = 13.5$

3 $S_3 = \frac{u_1\left(1-r^3\right)}{1-r} = 13 \rightarrow \frac{1-r^3}{13} = \frac{1-r}{u_1}$ (1)

$S_\infty = \frac{u_1}{1-r} = \frac{27}{2} \rightarrow \frac{2}{27} = \frac{1-r}{u_1}$ (2)

Equating (1) and (2)

$\frac{1-r^3}{13} = \frac{2}{27} \rightarrow 1 - r^3 = \frac{26}{27}$

$r^3 = \frac{1}{27} \rightarrow r = \frac{1}{3}$

put $r = \frac{1}{3}$ in S_∞

$\frac{u_1}{\left(1-\frac{1}{3}\right)} = \frac{27}{2} \rightarrow u_1 = 9$

$S_5 = \frac{9\left(1-\left(\frac{1}{3}\right)^5\right)}{\left(1-\frac{1}{3}\right)} = 13.5\left(1-\left(\frac{1}{3}\right)^5\right)$

$S_5 = 13.\overline{4}$

4 $u_3r^3 = u_6$, so $24r^3 = 3 \rightarrow r^3 = \frac{1}{8} \rightarrow r = \frac{1}{2}$

since $u_1r^2 = u_3$, $u_1\left(\frac{1}{2}\right)^2 = 24 \rightarrow u_1 = 96$

$S_\infty = \frac{96}{\left(1-\frac{1}{2}\right)} = 192$

5 $u_2 = u_1(r) = 12 \rightarrow r = \frac{12}{u_1}$ (1)

$S_\infty = \frac{u_1}{1-r} = 64 \rightarrow 1 - r = \frac{u_1}{64} \rightarrow r = 1 - \frac{u_1}{64}$ (2)

$(1) = (2) \Rightarrow 1 - \frac{u_1}{64} = \frac{12}{u_1} \rightarrow u_1 - \frac{(u_1)^2}{64} = 12$

$\rightarrow 64u_1 - (u_1)^2 = 768$

$(u_1)^2 - 64u_1 + 768 = 0$

$u_1 = 16$ *or* 48

6 $r = 0.4$, $S_\infty = \frac{u_1}{1-r} = 250$

$\frac{u_1}{1-0.4} = 250$

$u_1 = 150$

7 $S_5 = \frac{u_1\left(1-r^5\right)}{1-r} = 3798$ (1)

$S_\infty = \frac{u_1}{1-r} = 4374 \rightarrow \frac{u_1\left(1-r^5\right)}{1-r} = 4374\left(1-r^5\right)$ (2)

$(1) = (2) \Rightarrow 4374(1 - r^5) = 3798$

$4374r^5 = 576$

$r^5 = \frac{576}{4374} = \frac{32}{243} \rightarrow r = \frac{2}{3}$

$S_\infty = \frac{u_1}{\left(1-\left(\frac{2}{3}\right)\right)} = 4374 \rightarrow u_1 = 1458$

$S_7 = \frac{1458\left(1-\left(\frac{2}{3}\right)^7\right)}{\left(1-\left(\frac{2}{3}\right)\right)} = 4374\left(1-\left(\frac{2}{3}\right)^7\right)$

$S_7 = 4118$

Exercise 6L

1 $u_6 = u_1 + 5d$, $u_4 = u_1 + 3d$. Since $u_6 = 3u_4$, we have $u_1 + 5d = 3(u_1 + 3d) \rightarrow u_1 + 5d = 3(u_1 + 9d)$

$\rightarrow 2u_1 + 4d = 0$

$u_8 = u_1 + 7d = 50$

$u_1 = -20$

Use simultaneous equation solver on GDC.

2 **a** $u_1 = 20$

$u_5 = u_1 + 4\text{d} = 12 + 4d = 15 \rightarrow 4d = 3 \rightarrow$

$d = 0.75$

$u_{20} = 12 + 19(0.75) = 26.25$

b $12 + (n - 1)(0.75) = 100 \rightarrow 0.75n - 0.75 = 88$

$0.75n = 88.75 \rightarrow n = 118.\overline{3}$

$n = 119$

3 **a** $2500(1.06)^8$
$3984.62

b $2500(1.015)^{32}$
$4025.81

c $2500(1.005)^{96}$
$4035.36

4 On GDC, let $Y1 = 12x - 7$, and
let $Y2 = 0.3(1.2)^{x-1}$
Using table, when x is 41, $Y1 = 485$,
and $Y2 \approx 440.93$
When x is 42, $Y1 = 497$, and $Y2 \approx 529.12$
$n = 42$

5 For arithmetic sequence, $S_n = \frac{n}{2}(75 + (n-1)100)$.
For geometric sequence, $S_n = \frac{6(1.5^n - 1)}{1.5 - 1}$.
On GDC, let $Y1 = \frac{6(1.5^x - 1)}{1.5 - 1}$, and
let $Y2 = \frac{x}{2}(2(75) + 100(x-1))$
Using table, when x is 17, $Y1 \approx 11\,811$,
and $Y2 = 14\,875$
When x is 18, $Y1 \approx 17\,723$, and $Y2 = 16\,650$
$n = 18$

6 $200(1.05)^3 \approx 232$

7 $275\,000(1.031)^n = 500\,000$
$(1.031)^n = \frac{20}{11}$

Use logarithms or other GDC methods.

$n \approx 19.6$ years

8 **a** $S_1 = 3(1)^2 - 2(1) = 1$
$S_2 = 3(2)^2 - 2(2) = 8$
$S_3 = 3(3)^2 - 2(3) = 21$

b $u_1 = S_1 = 1$
$u_2 = S_2 - S_1 = 8 - 1 = 7$
$u_3 = S_3 - S_2 = 21 - 8 = 13$

c $d = 6$
$u_n = 1 + (n-1)(6) = 6n - 5$

9 **a** $S_1 = 2^{1+2} - 4 = 4$
$S_2 = 2^{2+2} - 4 = 12$
$S_3 = 2^{3+2} - 4 = 28$

b $u_1 = S_1 = 4$
$u_2 = S_2 - S_1 = 12 - 4 = 8$
$u_3 = S_3 - S_2 = 28 - 12 = 16$

c $r = \frac{u_2}{u_1} = \frac{8}{4} = 2$

$S_n = \frac{4(2^n - 1)}{2 - 1} = 4(2^{n-1})$

10 After n months, species A has $1200(1.0125)^n$ spiders, species B has $50000 - (175)n$ spiders.
On GDC, let $Y1 = 12\,000(1.0125)^x$, and let
$Y2 = 50\,000 - 175x$.
Using graph (intersect) or solver ($Y1 - Y2 = 0$),
$x \approx 86.039$.
approx. 86 months
After 10 years Mohiro has 3000×1.03^{10}
$= \$4031.75$
After 10 years Ryan has $3000\left(1 + \frac{0.03}{12}\right)^{120}$
$= \$4048.06$
So Ryan has $16.31 more than Mohira

Exercise 6M

1 $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{6(2)} = 10$

2 $\binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{40320}{2(720)} = 28$

3 $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{5040}{6(24)} = 35$

4 $\binom{9}{6} = \frac{9!}{6!(9-6)!} = \frac{362880}{720(6)} = 84$

5 $\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{720}{24(2)} = 15$

6 $\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{3628800}{6(5040)} = 120$

Exercise 6N

1 $\binom{5}{0}(y)^5(3)^0 + \binom{5}{1}(y)^4(3)^1 + \binom{5}{2}(y)^3(3)^2 + \binom{5}{3}(y)^2(3)^3 + \binom{5}{4}(y)^1(3)^4 + \binom{5}{5}(y)^0(3)^5$

$(1)(y^5)(1) + (5)(y^4)(3) + (10)(y^3)(9) + (10)(y^2)27 + (5)(y)(81) + (1)(1)(243)$
$y^5 + 15y^4 + 90y^3 + 270y^2 + 405y + 243$

2 $\binom{4}{0}(2b)^4(-1)^0 + \binom{4}{1}(2b)^3(-1)^1 + \binom{4}{2}(2b)^2(-1)^2 + \binom{4}{3}(2b)^1(-1)^3 + \binom{4}{4}(2b)^0(-1)^4$

$(1)(16b^4)(1) + (4)(8b^3)(-1) + (6)(4b^2)(1) + (4)(2b)(-1) + (1)(1)(1)$
$16b^4 - 32b^3 + 24b^2 - 8b + 1$

3 $\binom{6}{0}(3a)^6(2)^0 + \binom{6}{1}(3a)^5(2)^1 + \binom{6}{2}(3a)^4(2)^2 + \binom{6}{3}(3a)^3(2)^3 + \binom{6}{4}(3a)^2(2)^4 + \binom{6}{5}(3a)^1(2)^5 + \binom{6}{6}(3a)^0(2)^6$

$(1)(729a^6)(1) + (6)(243a^5)(2) + (15)(81a^4)(4) + (20)(27a^3)(8) + (15)(9a^2)(16) + (6)(3a)(32) + (1)(1)(64)$
$729a^6 + 2916a^5 + 4860a^4 + 4320a^3 + 2160a^2 + 576a + 64$

4 $\binom{3}{0}(x^2)^3\left(\frac{2}{x}\right)^0 + \binom{3}{1}(x^2)^2\left(\frac{2}{x}\right)^1 + \binom{3}{2}(x^2)^1\left(\frac{2}{x}\right)^2 + \binom{3}{3}(x^2)^0\left(\frac{2}{x}\right)^3$

$$(1)(x^6)(1)+(3)(x^4)\left(\frac{2}{x}\right)+(3)(x^2)\left(\frac{4}{x^2}\right)+(1)(1)\left(\frac{8}{x^3}\right)$$

$$x^6+6x^3+12+\frac{8}{x^3}$$

5 $\binom{8}{0}x^8y^0+\binom{8}{1}x^7y^1+\binom{8}{2}x^6y^2+\binom{8}{3}x^5y^3+\binom{8}{4}x^4y^4+\binom{8}{5}x^3y^5+\binom{8}{6}x^2y^6+\binom{8}{7}x^1y^7+\binom{8}{8}x^0y^8$

$x^8 + 8x^7 y + 28x^6 y^2 + 56x^5 y^3 + 70x^4 y^4 + 56x^3 y^5 + 28x^2 y^6 + 8xy^7 + y^8$

6 $\binom{4}{0}(3a)^4(-2b)^0+\binom{4}{1}(3a)^3(-2b)^1+\binom{4}{2}(3a)^2(-2b)^2+\binom{4}{3}(3a)^1(-2b)^3+\binom{4}{4}(3a)^0(-2b)^4$

$(1)(81a^4)(1) + (4)(27a^3)(-2b) + (6)(9a^2)(4b^2) + (4)(3a)(-8b^3) + (1)(1)(16b^4)$

$81a^4 -216a^3b + 216a^2b^2 -96ab^3 + 16b^4$

7 $\binom{5}{0}(3c)^5\left(\frac{2}{d}\right)^0+\binom{5}{1}(3c)^4\left(\frac{2}{d}\right)^1+\binom{5}{2}(3c)^3\left(\frac{2}{d}\right)^2+\binom{5}{3}(3c)^2\left(\frac{2}{d}\right)^3+\binom{5}{4}(3c)^1\left(\frac{2}{d}\right)^4+\binom{5}{5}(3c)^0\left(\frac{2}{d}\right)^5$

$(1)(243c^5)(1) + (5)(81c^4)\left(\frac{2}{d}\right) + (10)(27c^3)\left(\frac{4}{d^2}\right) + (10)(9c^2)\left(\frac{8}{d^3}\right) + (5)(3c)\left(\frac{16}{d^4}\right)^4 + (1)(1)\left(\frac{32}{d^5}\right)$

$243c^5 + \frac{810c^4}{d} + \frac{1080c^3}{d^2} + \frac{720c^2}{d^3} + \frac{240c}{d^4} + \frac{32}{d^5}$

8 $\binom{3}{0}(4x^2)^3\left(\frac{1}{2y}\right)^0+\binom{3}{1}(4x^2)^2\left(\frac{1}{2y}\right)^1+\binom{3}{2}(4x^2)^1\left(\frac{1}{2y}\right)^2+\binom{3}{3}(4x^2)^0\left(\frac{1}{2y}\right)^3$

$(1)(64x^6)(1) + (3)(16x^4)\left(\frac{1}{2y}\right) + (3)(4x^2)\left(\frac{1}{4y^2}\right) + (1)(1)\left(\frac{1}{8y^3}\right)$

$64x^6 + \frac{24x^4}{y} + \frac{3x^2}{y^2} + \frac{1}{8y^3}$

Exercise 6O

1 $\binom{7}{2}(x)^5(-4)^2 = (21)(x^5)(16) = 336x^5$

2 $\binom{5}{1}(4y)^4(-1)^1=(5)(256y^4)(-1)=-1280y^4$

3 $\binom{6}{4}(2a)^2(-3b)^4=(15)(4a^2)(81b^4)=4860a^2b^4$

4 $\binom{9}{9}(x)^0(-2)^9 = (1)(1)(-512)=-512$

5 $\binom{6}{3}(px)^3(1)^3=(20)(p^3x^3)(1)=20p^3x^3$

$20p^3 x^3 = 160x^3$

$20p^3 = 160 \rightarrow p^3 = 8$

$p = 2$

6 $\binom{7}{2}(3x)^5(q)^2=(21)(243x^5)(q^2)=5103x^5q^2$

$5103q^2x^5 = 81648x^5$

$5103q^2 = 81648 \rightarrow q^2 = 16$

$q = \pm4$

7 $\binom{8}{4}(4x)^4\left(\frac{1}{x}\right)^4 = (70)(256x^4)\left(\frac{1}{x^4}\right) = 17920$

8 $\binom{6}{4}(2x^2)^2\left(-\frac{3}{x}\right)^4=(15)(4x^4)\left(\frac{81}{x^4}\right) = 4860$

9 $\binom{n}{3}(x)^3(1)^{n-3}=\binom{n}{3}x^3$

$\binom{n}{2}(x)^2(1)^{n-2}=\binom{n}{2}x^2$

$\binom{n}{3}=\frac{n(n-1)(n-2)(n-3)(n-4)\dots(1)}{3!(n-3)(n-4)\dots(1)}=\frac{n(n-1)(n-2)}{3!}=\frac{n(n-1)(n-2)}{6}$

$\binom{n}{2}=\frac{n(n-1)(n-2)(n-3)\dots(1)}{2!(n-2)(n-3)\dots(1)}=\frac{n(n-1)}{2!}=\frac{n(n-1)}{2}$

$\binom{n}{3}=2\binom{n}{2} \rightarrow \frac{n(n-1)(n-2)}{6}=2\left(\frac{n(n-1)}{2}\right)$

$\frac{n(n-1)(n-2)}{6} = n(n-1) \rightarrow n(n-1)(n-2) = 6n(n-1)$

$n - 2 = 6$

$n = 8$

10 $\binom{n}{3}(x)^3(2)^{n-3}=\binom{n}{3}\left(\frac{2^n}{8}\right)x^3$

$\binom{n}{4}(x)^4(2)^{n-4}=\binom{n}{4}\left(\frac{2^n}{16}\right)x^4$

$\binom{n}{3}=\frac{n(n-1)(n-2)(n-3)(n-4)\dots(1)}{3!(n-3)(n-4)\dots(1)}=\frac{n(n-1)(n-2)}{3!}=\frac{n(n-1)(n-2)}{6}$

$\binom{n}{4}=\frac{n(n-1)(n-2)(n-3)(n-4)\dots(1)}{4!(n-4)(n-5)\dots(1)}=\frac{n(n-1)(n-2)(n-3)}{4!}=\frac{n(n-1)(n-2)(n-3)}{24}$

$\binom{n}{3}\left(\frac{2^n}{8}\right)=2\binom{n}{4}\left(\frac{2^n}{16}\right)\rightarrow\left(\frac{n(n-1)(n-2)}{6}\right)\left(\frac{2^n}{8}\right)=2\left(\frac{n(n-1)(n-2)(n-3)}{24}\right)\left(\frac{2^n}{16}\right)$

$\left(\frac{n(n-1)(n-2)}{48}\right)(2^n)=\left(\frac{n(n-1)(n-2)(n-3)}{192}\right)(2^n)$

$\left(\frac{n(n-1)(n-2)}{48}\right)=\left(\frac{n(n-1)(n-2)(n-3)}{192}\right)$

$\rightarrow 4(n(n-1)(n-2)) = (n(n-1)(n-2)(n-3))$

$n - 3 = 4$

$n = 7$

Review exercise

1 **a** $d = u_2 - u_1 = 7 - 3 = 4$

b $u_n = 3 + 70(4) = 283$

c $3 + 4(n-1) = 99 \rightarrow 4n - 1 = 99$

$4n = 100$

$n = 25$

2 **a** $r = \frac{u_2}{u_1} = \frac{16}{64} = \frac{1}{4}$

b $u_4 = 64\left(\frac{1}{4}\right)^3 = 1$

c $S_\infty = \frac{64}{\left(1-\frac{1}{4}\right)} = \frac{64}{\left(\frac{3}{4}\right)} = \frac{256}{3}$

3 **a** $u_6 = u_1 + 5d = 25$

$u_{12} = u_1 + 11d = 49$

$6d = 24$

$d = 4$

Subtract the first equation from the second.

b $u_1 + 5(4) = 25$

$u_1 = 5$

4 **a** $u_3 = 22 + 2d = 38 \rightarrow 2d = 16 \rightarrow d = 8$

$u_2 = x = 22 + 8 = 30$

b $u_{31} = 22 + 30(8) = 262$

5 $\sum_{a=1}^{4}(3^a) = 3^1 + 3^2 + 3^3 + 3^4 = 3 + 9 + 27 + 81 = 120$

6 **a** $\frac{1}{4}$

b $S_\infty = \frac{800}{\left(1-\frac{1}{4}\right)} = \frac{800}{\left(\frac{3}{4}\right)} = \frac{3200}{3}$

7 Since $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$, $\frac{12}{x} = \frac{9x}{12} \rightarrow 9x^2 = 144$

$x^2 = 16$

$x = \pm 4$

8 $\binom{5}{2}(2x)^3(3)^2 = 10(8x^3)(9) = 720x^3$

9 **a** $u_1 = 3,\ d = 2,\ 3 + 2(n-1) = 35 \rightarrow$

$2n + 1 = 35 \rightarrow 2n = 34$

$n = 17$

b $S_{17} = \frac{17}{2}(2(3) + 16(2)) = 8.5(6 + 32) = 323$

Review exercise

1 **a** $S_{25} = \frac{25}{2}(2(4) + 24(d)) = 1000$

$12.5(8 + 24d) = 100 + 300d = 1000$

$300d = 900$

$d = 3$

b $u_{17} = 4 + 16(3) = 52$

2 **a** $u_{63} = 3 + 62(1.5) = 96$

b $\frac{n}{2}(2(3) + (n-1)(1.5)) = 840$

$\frac{n}{2}(1.5n + 4.5) = 840$

$1.5n^2 + 4.5n = 1680$

$1.5n^2 + 4.5n - 1680 = 0$

$n = 32$

Use quadratic equation solver on GDC.

3 **a** $u_{10} = u_1 + 9d = 25$

$S_{10} = \frac{10}{2}(2u_1 + 9d) = 160$

$= \frac{10}{2}(u_1 + [u_1 + 9d]) = 160$

using $u_1 + 9d = 25$, we see

$\frac{10}{2}(u_1 + 25) = 160 \rightarrow 5u_1 + 125 = 160$

$5u_1 = 35$

$u_1 = 7$

$7 + 9d = 25$

$9d = 18$

$d = 2$

b $S_{24} = \frac{24}{2}(2(7) + 23(2)) = 720$

4 **a** $u_1 = 3,\ u_6 = u_1r^5 = 96.$

$3r^5 = 96$

$r^5 = 32$

$r = 2$

b $u_n = 3(2^{n-1})$, so $3(2^{n-1}) > 3000$

$2^{n-1} > 1000$

$n - 1 > 9.97$

$n > 10.97$

$n = 11$

5 arithmetic sequence: $u_n = 28 + 50(n-1)$

geometric sequence: $u_n = 1(1.5)^{n-1}$

On GDC, let $y_1 = 28 + 50(x-1)$ and $y_2 = (1.5)^{x-1}$.

Using table, we see y_2 becomes bigger than y_1 when $n = 18$.

6 $u_3 = u_1(r^2) = 45 \rightarrow u_1 = \frac{45}{r^2}$

$S_7 = \frac{u_1(1-r^7)}{1-r} = 2735 \rightarrow u_1 = \frac{2735(1-r)}{(1-r^7)}$

$\frac{45}{r^2} = \frac{2735(1-r)}{(1-r^7)} \rightarrow 45(1-r^7) = 2735r^2(1-r)$

$45 - 45r^7 = 2735r^2 - 2735r^3$

$45r^7 - 2735r^3 + 2735r^2 - 45 = 0$

$r = -3$

$u_1 = \frac{45}{(-3)^2} = 5$

7 $\binom{7}{3}\left(\frac{x}{2}\right)^4(-3)^3 = 35\left(\frac{x^4}{16}\right)(-27) = \frac{-945x^4}{16}$

8 $\binom{8}{3}(ax)^5(2)^3 = 56(a^5x^5)(8) = 448a^5x^5 = \frac{7}{16}x^5$

$448a^5 = \frac{7}{16}$

$a^5 = \frac{7}{7168} = \frac{1}{1024}$

$a = \frac{1}{4}$

9 **a** In 2040, $n = 30$.

$3.4(1.016)^{30} \approx 5.4738$

approx. 5.47 million

b $3.4(1.016)^n = 7$

$(1.016)^n = \frac{7}{3.4}$

$n \approx 45.49$

the year will be 2055

7 Solutions

Answers

Skills check

1 **a** $9x^4 + 15x^3 + 3x$

$= 3x(3x^3 + 5x^2 + 1)$

b $4x^2 - 9 = (2x + 3)(2x - 3)$

Remember this is the difference of two squares.

c $x^2 - 5x + 6 = x^2 - 2x - 3x + 6$

$= x(x - 2) - 3(x - 2)$

$= (x - 3)(x - 2)$

d $2x^2 - 9x - 5 = 2x^2 + x - 10x - 5$

$= x(2x + 1) - 5(2x + 1)$

$= (x - 5)(2x + 1)$

2 **a** $(x + 2)^3 = 1(x)^3(2)^0 + 3(x)^2(2)^1 + 3(x)^1(2)^2 + 1(x)^0(2)^3$

$= x^3 + 6x^2 + 12x + 8$

b $(3x - 1)^4 = 1(3x)^4(-1)^0 + 4(3x)^3(-1)^1 + 6(3x)^2(7)^2 + 4(3x)^1(-1)^3 + 1(3x)^0(-1)^4$

$= 81x^4 + 108x^3(-1) + 54x^2 + 12x(-1) + 1$

$= 81x^4 - 108x^3 + 54x^2 - 12x + 1$

c $(2x + 3y)^3 = 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

3 **a** $\frac{1}{x^6} = x^{-6}$

b $\frac{4}{x^3} = 4x^{-3}$

c $5\sqrt{x} = 5x^{\frac{1}{2}}$

d $\sqrt[7]{x^5} = (x^5)^{\frac{1}{7}} = x^{\frac{5}{7}}$

e $\frac{7}{\sqrt{x^3}} = \frac{7}{(x^3)^{\frac{1}{2}}} = 7(x^3)^{-\frac{1}{2}} = 7x^{\frac{-3}{2}}$

Exercise 7A

1 1, 3, 5, 7, …

The terms in the sequence keep getting larger without bound, therefore the series is divergent.

2 3.49, 3.499, 3.499, 3.4999, …

The terms in the sequence get closer and closer to 3.5. The sequence is convergent and the limit is 3.5

3 $\frac{1}{10}, -\frac{1}{100}, \frac{1}{1000}, -\frac{1}{10,000}, \ldots$

Although the terms are alternating between positive and negative, they are getting closer and closer to zero. The sequence is convergent and the limit is 0.

4 $\frac{20}{27}, \frac{121}{162}, \frac{182}{243}, \frac{1093}{1458}, \frac{1640}{2187} \ldots$

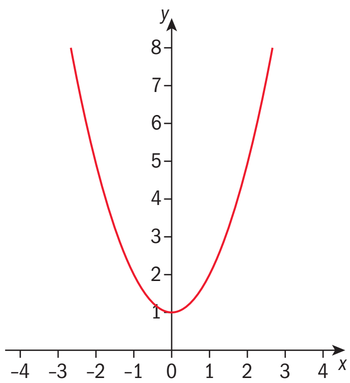
Look at the decimal value of the terms. To five decimal places we have: 0.74074, 0.74691, 0.74897, 0.74966, 0.74989. The terms in the sequence get closer and closer to 0.75. The sequence is convergent and the limit is 0.75

5 3, 4, 3, 4, 3, 4, …

The terms are alternating between 3 and 4 without getting closer to a fixed value. The sequence is divergent.

Exercise 7B

1 $\lim_{x\to 3}(x^2 + 1)$

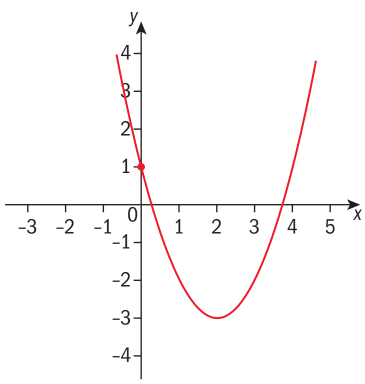


x	$f(x)$
2.5	7.2500
2.6	7.7600
2.7	8.2900
2.8	8.8400
2.9	9.4100
3.0	10.0000
3.1	10.6100
3.2	11.2400
3.3	11.8900
3.4	12.5600
3.5	13.2500

$f(x) = x^2 + 1$ approaches 10 as x approaches 3 from above and below. Therefore $\lim_{x\to 3}(x^2 + 1)$ exists, and $\lim_{x\to 3}(x^2 + 1) = 3^2 + 1$

$= 10$

2 $\lim_{x\to 0}\frac{x^3 - 4x^2 + x}{x}$



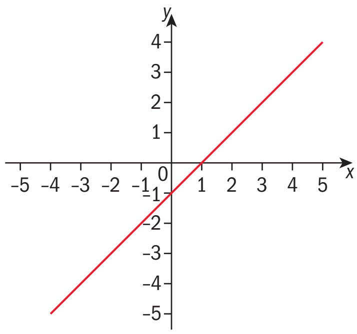
x	$f(x)$
−0.5	3.2500
−0.4	2.7600
−0.3	2.2900
−0.2	1.8400
−0.1	1.4100
0	
0.1	0.6100
0.2	0.2400
0.3	−0.1100
0.4	−0.4400
0.5	−0.7500

Although the function is undefined at 0 it has a limit of 1 at 0

$f(x) = \frac{x^3 - 4x + x}{x}$ approches 1 as x approaches 0, from above and below. Thus, $\lim_{x\to 0}\frac{x^3 - 4x + x}{x}$ exits, and

$$\lim_{x\to 0}\frac{x^3 - 4x^2 + x}{x} = \lim_{x\to 0}(x^2 - 4x + 1)$$
$$= 0^2 - 4(0) + 1$$
$$= 1$$

3 $\lim_{x\to 2}\frac{x^2 - 3x + 2}{x - 2}$



x	f(x)
1.5	0.5000
1.6	0.6000
1.7	0.7000
1.8	0.8000
1.9	0.9000
2.0	
2.1	1.1000
2.2	1.2000
2.3	1.3000
2.4	1.4000
2.5	1.5000

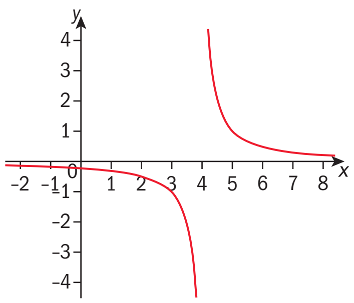
Although the function is undefined at 2 it has a limit of 0 at 2

$f(x) = \frac{x^2 - 3x + 2}{x - 2}$ approaches 1 as x approaches 2 from above and below.

Thus, $\lim_{x\to 2}\frac{x^2 - 3x + 2}{x - 2}$ exists, and

$$\lim_{x\to 2}\frac{x^2 - 3x + 2}{x - 2} = \lim_{x\to 2}\frac{(x - 2)(x - 1)}{x - 2} = \lim_{x\to 2}(x - 1) = 2 - 1 = 1$$

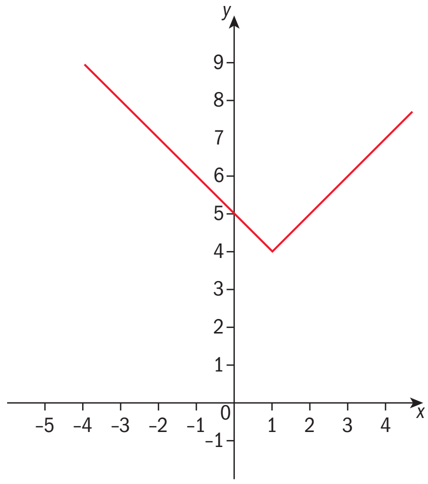
4 $\lim_{x\to 4}\frac{1}{x - 4}$



x	f(x)
3.5	−2.0000
3.6	−2.5000
3.7	−3.3333
3.8	−5.0000
3.9	−10.0000
4.0	
4.1	10.0000
4.2	5.0000
4.3	3.3333
4.4	2.5000
4.5	2.0000

$\lim_{x\to 4}\frac{1}{x - 4}$ does not exist at 4 since the function approaches $-\infty$ from the left side of 4 and ∞ on the right side of 4.

5 $\lim_{x\to 1} f(x)$; where $f(x) = \begin{cases} x + 3 & \text{for } x \geq 1 \\ -x + 5 & \text{for } x < 1 \end{cases}$

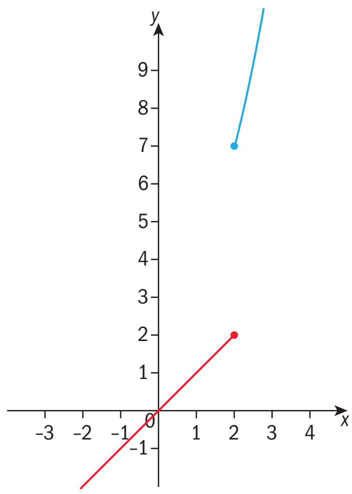


x	f(x)
0.5	4.5000
0.6	4.4000
0.7	4.3000
0.8	4.2000
0.9	4.1000
1.0	4.0000

x	f(x)
1.0	4.0000
1.1	4.1000
1.2	4.2000
1.3	4.3000
1.4	4.4000
1.5	4.5000

$\lim_{x\to 1} f(x) = 4$ since $f(x)$ approaches 4 as x gets close to 1 from either side.

6 $\lim_{x\to 2} f(x)$; where $f(x) = \begin{cases} x^2 + 3 & \text{for } x \geq 2 \\ x & \text{for } x < 2 \end{cases}$



x	f(x)
1.5	1.5000
1.6	1.6000
1.7	1.7000
1.8	1.8000
1.9	1.9000

x	f(x)
2.0	7.0000
2.1	7.4100
2.2	7.8400
2.3	8.2900
2.4	8.7600
2.5	9.2500

f approaches 2 as x approaches 2 from the left and f approaches 7 as x approaches 2 from the right. So $\lim_{x\to 2} f(x)$ does not exist.

Exercise 7C

1 $f(x) = 3x + 4$

$$\frac{f(x+h) - f(x)}{h} = \frac{[3(x+h)+4] - (3x+4)}{h}$$
$$= \frac{3x + 3h + 4 - 3x - 4}{h} = \frac{3h}{h} = 3$$

2 $f(x)=2x^2-1$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\left[2(x+h)^2-1\right]-\left(2x^2-1\right)}{h}\\&=\frac{\left[2(x^2+2xh+h^2)-1\right]-\left(2x^2-1\right)}{h}\\&=\frac{2x^2+4xh+2h^2-1-2x^2+1}{h}\\&=\frac{4xh+2h^2}{h}\\&=\frac{h(4x+2h)}{h}\\&=4x+2h\end{aligned}$$

3 $f(x)=x^2+2x+3$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\left[(x+h)^2+2(x+h)+3\right]-\left(x^2+2x+3\right)}{h}\\&=\frac{\left[(x^2+2xh+h^2)+(2x+2h)+3\right]-\left(x^2+2x+3\right)}{h}\\&=\frac{x^2+2xh+h^2+2x+2h+3-x^2-2x-3}{h}\\&=\frac{2xh+h^2+2h}{h}\\&=\frac{h(2x+h+2)}{h}\\&=2x+h+2\end{aligned}$$

Exercise 7D

1 $f(x)=2x-3$; $x=2$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\left[2(x+h)-3\right]-(2x-3)}{h}\\&=\lim_{h\to0}\frac{2x+2h-3-2x+3}{h}\\&=\lim_{h\to0}\frac{2h}{h}\\&=\lim_{h\to0}2\\f'(x)&=2\\m&=f'(2)\\&=2\end{aligned}$$

2 $f(x)=3x^2+2x$; $x=-3$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\left[3(x+h)^2+2(x+h)\right]-\left(3x^2+2x\right)}{h}\\&=\lim_{h\to0}\frac{\left[3(x^2+2xh+h^2)+2(x+h)\right]-\left(3x^2+2x\right)}{h}\\&=\lim_{h\to0}\frac{3x^2+6xh+3h^2+2x+2h-3x^2-2x}{h}\\&=\lim_{h\to0}\frac{6xh+3h^2+2h}{h}\\&=\lim_{h\to0}(6x+3h+2)\\&=6x+3(0)+2\\f'(x)&=6x+2\\m&=f'(-3)\\&=6(-3)+2\\&=-16\end{aligned}$$

3 $f(x)=x^2-x+2$; $x=1$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\left[(x+h)^2-(x+h)+2\right]-\left(x^2-x+2\right)}{h}\\&=\lim_{h\to0}\frac{\left[(x^2+2xh+h^2)-(x+h)+2\right]-\left(x^2-x+2\right)}{h}\\&=\lim_{h\to0}\frac{x^2+2xh+h^2-x-h+2-x^2+x-2}{h}\\&=\lim_{h\to0}\frac{2xh+h^2-h}{h}\\&=\lim_{h\to0}(2x+h-1)\\&=2x+0-1\\f'(x)&=2x-1\\m&=f'(1)\\&=2(1)-1=1\end{aligned}$$

Exercise 7E

1 $f(x)=x^5$

$f'(x)=5x^{5-1}=5x^4$

2 $f(x)=x^8$

$f'(x)=8x^{8-1}=8x^7$

3
$$\begin{aligned}f(x)&=\frac{1}{x^4}\\&=x^{-4}\\f'(x)&=-4x^{-4-1}\\&=-4x^{-5}=-\frac{4}{x^5}\end{aligned}$$

4
$$\begin{aligned}f(x)&=\sqrt[3]{x}\\&=x^{\frac{1}{3}}\\f'(x)&=\frac{1}{3}x^{\frac{1}{3}-1}\\&=\frac{1}{3}x^{-\frac{2}{3}}\\&=\frac{1}{3x^{\frac{2}{3}}}\text{ or }\frac{1}{3\sqrt[3]{x^2}}\end{aligned}$$

5
$$\begin{aligned}f(x)&=\frac{1}{\sqrt{x}}\\&=x^{-\frac{1}{2}}\\f'(x)&=-\frac{1}{2}x^{-\frac{1}{2}-1}\\&=-\frac{1}{2}x^{-\frac{3}{2}}\\&=-\frac{1}{2x^{\frac{3}{2}}}\text{ or }-\frac{1}{2\sqrt{x^3}}\end{aligned}$$

6
$$\begin{aligned}f(x)&=\sqrt[5]{x^3}\\&=x^{\frac{3}{5}}\\f'(x)&=\frac{3}{5}x^{\frac{3}{5}-1}\\&=\frac{3}{5}x^{-\frac{2}{5}}\\&=\frac{3}{5x^{\frac{2}{5}}}\text{ or }\frac{3}{5\sqrt[5]{x^2}}\end{aligned}$$

Exercise 7F

1 $f(x)=\frac{2}{x^8}$
$=2x^{-8}$
$f'(x)=2(-8x^{-8-1})$
$=-16x^{-9}$
$=-\frac{16}{x^9}$

2 $f(x)=5$
$f'(x)=0$

3 $f(x)=x^3-\frac{3}{x^2}$
$=x^3-3x^{-2}$
$f'(x)=3x^{3-1}-3(-2x^{-2-1})$
$=3x^2+6x^{-3}$
$=3x^2+\frac{6}{x^3}$ or $\frac{3x^5+6}{x^3}$

4 $f(x)=\pi x^5$
$f'(x)=\pi\left(5x^{5-1}\right)$
$=5\pi x^4$

5 $f(x)=(x-4)^2$
$=x^2-8x+16$
$f'(x)=2x^{2-1}-8(1x^{1-1})+0$
$=2x-8x^0$
$=2x-8$

6 $f(x)=\sqrt{x}-4\sqrt[3]{x}$
$=x^{\frac{1}{2}}-4x^{\frac{1}{3}}$
$f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}-4\left(\frac{1}{3}x^{\frac{1}{3}-1}\right)$
$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{2}{3}}$
$=\frac{1}{2x^{\frac{1}{2}}}-\frac{4}{3x^{\frac{2}{3}}}$ or $\frac{1}{2\sqrt{x}}-\frac{4}{3\sqrt[3]{x^2}}$

7 $f(x)=\frac{3}{4x^2}$
$=\frac{3}{4}x^{-2}$
$f'(x)=\frac{3}{4}(-2x^{-2-1})$
$=-\frac{3}{2}x^{-3}$
$=-\frac{3}{2x^3}$

8 $f(x)=\frac{3}{(4x)^2}$
$=\frac{3}{16x^2}$
$=\frac{3}{16}x^{-2}$
$f'(x)=\frac{3}{16}(x^{-2})$
$=\frac{3}{16}(-2x^{-2-1})$
$=-\frac{3}{8}x^{-3}$
$=-\frac{3}{8x^3}$

9 $f(x)=12-x^4$
$f'(x)=0-4x^{4-1}$
$=-4x^3$

10 $f(x)=\sqrt{x}\left(\sqrt[3]{x}+\sqrt[4]{x}\right)$
$=x^{\frac{1}{2}}\left(x^{\frac{1}{3}}+x^{\frac{1}{4}}\right)$
$=x^{\frac{5}{6}}+x^{\frac{3}{4}}$
$f'(x)=\frac{5}{6}\left(x^{\frac{5}{6}-1}\right)+\frac{3}{4}\left(x^{\frac{3}{4}-1}\right)$
$=\frac{5}{6}x^{-\frac{1}{6}}+\frac{3}{4}x^{-\frac{1}{4}}$
$=\frac{5}{6x^{\frac{1}{6}}}+\frac{3}{4x^{\frac{1}{4}}}$ or $\frac{5}{6\sqrt[6]{x}}+\frac{3}{4\sqrt[4]{x}}$

11 $f(x)=3x^4-2x^2+5$
$f'(x)=3\left(4x^{4-1}\right)-2\left(2x^{2-1}\right)+0$
$=12x^3-4x$

12 $f(x)=2x^2+3x+7$
$f'(x)=2\left(2x^{2-1}\right)+3\left(1x^{1-1}\right)+0$
$=4x+3x^0$
$=4x+3$

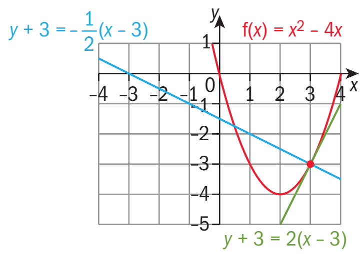
13 $f(x)=x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1$
$f'(x)=\frac{2}{3}x^{\frac{2}{3}-1}+2\left(\frac{1}{3}x^{\frac{1}{3}-1}\right)+0$
$=\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}x^{-\frac{2}{3}}$
$=\frac{2}{3x^{\frac{1}{3}}}+\frac{2}{3x^{\frac{2}{3}}}$ or $\frac{2}{3\sqrt[3]{x}}+\frac{2}{3\sqrt[3]{x^2}}$

14 $f(x)=2x(x^2-3x)$
$=2x^3-6x^2$
$f'(x)=2\left(3x^{3-1}\right)-6\left(2x^{2-1}\right)$
$=6x^2-12x$

15 $f(x)=(x^2+3x)(x-1)$
$=x^3+2x^2-3x$
$f'(x)=3x^{3-1}+2\left(2x^{2-1}\right)-3\left(1x^{1-1}\right)$
$=3x^2+4x-3x^0$
$=3x^2+4x-3$

Exercise 7G

1 $f(x)=x^2-4x$
$f'(x)=2x-4$
$m_{\text{tangent}}=f'(3)$
$=2(3)-4$
$=2$



$m_{\text{normal}}=-\frac{1}{2}$

tangent line: $y+3=2(x-3)$ or $y=2x-9$

normal line: $y+3=-\frac{1}{2}(x-3)$ or $y=-\frac{1}{2}x-\frac{3}{2}$

2 **a** $f(x) = x^2 + 2x + 1$ at $(-3, 4)$

$f'(x) = 2x + 2$

$m = f'(-3)$

$= 2(-3) + 2$

$= -4$

$y - 4 = -4(x + 3)$ or $y = -4x - 8$

b $f(x) = 2\sqrt{x} + 4$ at $x = 1$

$f(1) = 2\sqrt{1} + 4$

$= 6$

$f(x) = 2x^{\frac{1}{2}} + 4$

$f'(x) = \frac{1}{x^{\frac{1}{2}}}$

$m = f'(1)$

$= \frac{1}{1^{\frac{1}{2}}}$

$= 1$

$y - 6 = 1(x - 1)$ or $y = x + 5$

c $f(x) = \frac{x^2 + 6}{x}$ at $(3, 5)$

$= x + 6x^{-1}$

$f'(x) = 1 - \frac{6}{x^2}$

$m = f'(3)$

$= 1 - \frac{6}{3^2}$

$= \frac{1}{3}$

$y - 5 = \frac{1}{3}(x - 3)$ or $y = \frac{1}{3}x + 4$

d $f(x) = \sqrt[4]{x} + \frac{8}{\sqrt{x}}$ at $x = 1$

$f(1) = \sqrt[4]{1} + \frac{8}{\sqrt{1}}$

$= 9$

$f(x) = x^{\frac{1}{4}} + 8x^{-\frac{1}{2}}$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} + 8\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= \frac{1}{4x^{\frac{3}{4}}} - \frac{4}{x^{\frac{3}{2}}}$

$m = f'(1)$

$= \frac{1}{4(1)^{\frac{3}{4}}} - \frac{4}{(1)^{\frac{3}{2}}}$

$= -\frac{15}{4}$

$y - 9 = -\frac{15}{4}(x - 1)$ or $y = -\frac{15}{4}x + \frac{51}{4}$

3 **a** $f(x) = 2x^2 - x - 3$ at $(2, 3)$

$f'(x) = 4x - 1$

$m_{\text{tangent}} = f'(2)$

$= 4(2) - 1$

$= 7$

$m_{\text{normal}} = -\frac{1}{7}$

$y - 3 = -\frac{1}{7}(x - 2)$ or $y = -\frac{1}{7}x + \frac{23}{7}$

b $f(x) = \frac{4}{x} - \frac{1}{x^2}$ at $x = -1$

$f(-1) = \frac{4}{-1} - \frac{1}{(-1)^2}$

$= -5$

$f(x) = 4x^{-1} - x^{-2}$

$f'(x) = -\frac{4}{x^2} + \frac{2}{x^3}$

$m_{\text{tangent}} = f'(-1)$

$= -\frac{4}{(-1)^2} + \frac{2}{(-1)^3}$

$= -6$

$m_{\text{normal}} = \frac{1}{6}$

$y + 5 = \frac{1}{6}(x + 1)$ or $y = -\frac{1}{6}x - \frac{29}{6}$

c $f(x) = (2x + 1)^2$ at $(2, 25)$

$= 4x^2 + 4x + 1$

$f'(x) = 8x + 4$

$m_{\text{tangent}} = f'(2)$

$= 8(2) + 4$

$= 20$

$m_{\text{normal}} = -\frac{1}{20}$

$y - 25 = -\frac{1}{20}(x - 2)$ or $y = -\frac{1}{20}x + 25\frac{1}{10}$

d $f(x) = 2\sqrt[3]{x} - \frac{4}{x^2}$ at $x = 1$

$f(1) = 2\sqrt[3]{1} - \frac{4}{1^2}$

$= -2$

$f(x) = 2x^{\frac{1}{3}} - 4x^{-2}$

$f'(x) = \frac{2}{3x^{\frac{2}{3}}} + \frac{8}{x^3}$

$m_{\text{tangent}} = f'(1)$

$= \frac{2}{3(1)^{\frac{2}{3}}} + \frac{8}{1^3}$

$= \frac{26}{3}$

$m_{\text{normal}} = -\frac{3}{26}$

$y + 2 = -\frac{3}{26}(x - 1)$ or $y = -\frac{3}{26}x - \frac{23}{26}$

4 $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

vertical normal lines $\Rightarrow$ horizontal tangent lines

$\Rightarrow f'(x) = 3x^2 - 3 = 0$

$3x^2 - 3 = 0$ since horizontal tangent lines have zero gradients

$3(x-1)(x+1) = 0$

$x = 1, -1$

So the vertical normal lines are $x = 1$ and $x = -1$.

5 $f(x) = 2x^2 + kx - 3$

$f'(x) = 4x + k$. $f'(-1) = 1$, so

$4(-1) + k = 1$

$k = 5$

Exercise 7H

1 $f(x) = 4\ln x$

$f'(x) = 4\left(\frac{1}{x}\right)$

$= \frac{4}{x}$

2 $f(x) = e^x + \sqrt{x}$

$= e^x + x^{\frac{1}{2}}$

$f'(x) = e^x + \frac{1}{2}x^{-\frac{1}{2}}$

$= e^x + \frac{1}{2x^{\frac{1}{2}}}$

3 $f(x) = \ln e^{3x^4} + \ln x$

$= 3x^4 + \ln x$

$f'(x) = 12x^3 + \frac{1}{x}$

4 $f(x) = e^{\ln 4x^2} + 3x + 1$

$= 4x^2 + 3x + 1$

$f'(x) = 8x + 3$

5 $f(x) = 2e^x + \ln x$

$f'(x) = 2e^x + \frac{1}{x}$

6 $f(x) = 5e^x + 4\ln e^x$

$= 5e^x + 4x$

$f'(x) = 5e^x + 4$

7 $f(x) = 4e^x - 7$

First, evaluate $f(x)$ at $x = \ln 3$

$f(\ln 3) = 4e^{\ln 3} - 7$

$= 4(3) - 7$

$= 5$

so $f(x)$ passes through $(\ln 3, 5)$.

$f'(x) = 4e^x$

$m = f'(\ln 3)$

$= 4e^{\ln 3}$

$= 4(3)$

$= 12$

$y - 5 = 12(x - \ln 3)$

8 $f(x) = \ln\left(e^{x^2}\right)$

$= x^2$

$f'(x) = 2x$

$m_{\text{tangent}} = f'(-3)$

$= 2(-3)$

$= -6$

$m_{\text{normal}} = \frac{1}{6}$

$y - 9 = \frac{1}{6}(x + 3)$

9 $f(x) = \ln x$

$f(e) = \ln e$

$= 1$

so $f(x)$ passes through (e, 1).

$f'(x) = \frac{1}{x}$

$m = f'(e)$

$= \frac{1}{e}$

$y - 1 = \frac{1}{e}(x - e)$ or $y = \frac{x}{e}$

10 $f(x) = 2x^2 + e^{\ln x} - 3$

$= 2x^2 + x - 3$

$f(2) = 2(2^2) + 2 - 3$

$= 7$

so $f(x)$ passes through (2, 7).

$f'(x) = 4x + 1$

$m_{\text{tangent}} = f'(2)$

$= 4(2) + 1$

$= 9$

$m_{\text{normal}} = -\frac{1}{9}$

$y - 7 = -\frac{1}{9}(x - 2)$ or $y = -\frac{1}{9}x + \frac{65}{9}$

11 $f(x) = 2e^x - 5$

$f'(x) = 2e^x$

$f'(3) = 2e^3 \approx 40.2$

12 $f(x) = \sqrt[3]{x} + \ln x$

$= x^{\frac{1}{3}} + \ln x$

$f'(x) = \frac{1}{3x^{\frac{2}{3}}} + \frac{1}{x}$

$f'(8) = \frac{1}{3\left(8^{\frac{2}{3}}\right)} + \frac{1}{8}$

$= \frac{1}{12} + \frac{1}{8}$

$= \frac{5}{24} \approx 0.208$

Exercise 7I

1 $f(x) = \frac{x^2}{x-4}$

$f'(x) = \frac{(x-4)(2x) - (x^2)(1)}{(x-4)^2}$

$= \frac{2x^2 - 8x - x^2}{(x-4)^2}$

$= \frac{x^2 - 8x}{(x-4)^2}$

2 $f(x)=(2x^3+x^2+x)(x^2+1)$

$f'(x)=(2x^3+x^2+x)(2x)+(x^2+1)(6x^2+2x+1)$

$=(4x^4+2x^3+2x^2)+(6x^4+2x^3+7x^2+2x+1)$

$=10x^4+4x^3+9x^2+2x+1$

3 $f(x)=\frac{\ln x}{x}$

$f'(x)=\frac{(x)\left(\frac{1}{x}\right)-(\ln x)(1)}{x^2}$

$=\frac{1-\ln x}{x^2}$

4 $f(x)=e^x\ln x$

$f(x)=(e^x)\left(\frac{1}{x}\right)+(\ln x)(e^x)$

$=\frac{e^x}{x}+e^x\ln x$

5 $f(x)=\frac{x-2}{x+4}$

$f'(x)=\frac{(x+4)(1)-(x-2)(1)}{(x+4)^2}$

$=\frac{x+4-x+2}{(x+4)^2}$

$=\frac{6}{(x+4)^2}$

6 $f(x)=\frac{e^x}{e^x+1}$

$f'(x)=\frac{(e^x+1)(e^x)-(e^x)(e^x)}{(e^x+1)^2}$

$=\frac{e^{2x}+e^x-e^{2x}}{(e^x+1)^2}$

$=\frac{e^x}{(e^x+1)^2}$

7 $f(x)=e^x(5x^3+4x)$

$f'(x)=(e^x)(15x^2+4)+(5x^3+4x)(e^x)$

$=e^x(5x^3+15x^2+4x+4)$

8 $f(x)=\frac{2-x^2}{x^3+1}$

$f'(x)=\frac{(x^3+1)(-2x)-(2-x^2)(3x^2)}{(x^3+1)^2}$

$=\frac{(-2x^4-2x)-(6x^2-3x^4)}{(x^3+1)^2}$

$=\frac{x^4-6x^2-2x}{(x^3+1)^2}$

9 $f(x)=xe^x$

$f'(x)=(x)(e^x)+(e^x)(1)$

$=e^x(x+1)$

horizontal tangent $\Rightarrow f'(x)=0$

$e^k(k+1)=0$

$e^k=0$ or $k+1=0$

$k=-1$ since $e^k\neq 0$ for any $k\in\mathbb{R}$.

10 $f(x)=\frac{x+1}{x-1}$

$f'(x)=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$

$=\frac{x-1-x-1}{(x-1)^2}$

$=-\frac{2}{(x-1)^2}$

$x+2y=10\Rightarrow y=-\frac{1}{2}x+5\Rightarrow m=-\frac{1}{2}$

parallel lines have the same slope $\Rightarrow f'(x)=-\frac{1}{2}$

$-\frac{2}{(x-1)^2}=-\frac{1}{2}$

$(x-1)^2=4$

$x-1=\pm\sqrt{4}$

$x=1\pm 2$

$x=3,\ -1$

$f(3)=\frac{3+1}{3-1}=2;\ f(-1)=\frac{-1+1}{-1-1}=0$

$y-2=-\frac{1}{2}(x-3);\ \ y-0=-\frac{1}{2}(x+1)$

Exercise 7J

1 $f(x)=\frac{2x^3-5x}{3}$

$=\frac{1}{3}(2x^3-5x)$

$f'(x)=\frac{1}{3}(6x^2-5)$ or $2x^2-\frac{5}{3}$

2 $f(x)=(x^2-5)(x^2+5)$

$=x^4-25$

$f'(x)=4x^3$

3 $f(x)=2e^x(x^2)$

$f'(x)=(2e^x)(2x)+(x^2)(2e^x)$

$=4xe^x+2x^2e^x$

$=2xe^x(2+x)$

4 $f(x)=\frac{2e^x}{x^2}$

$f'(x)=\frac{(x^2)(2e^x)-(2e^x)(2x)}{(x^2)^2}$

$=\frac{2x^2e^x-4xe^x}{x^4}$

$=\frac{2xe^x-4e^x}{x^3}$

5 $f(x)=e^{\ln x^3}+\frac{4}{\sqrt[5]{x^4}}$

$=x^3+4x^{-\frac{4}{5}}$

$f'(x)=3x^2-\frac{16}{5}x^{-\frac{9}{5}}$

$=3x^2-\frac{16}{5x^{\frac{9}{5}}}$

6 $f(x)=\frac{x^2}{e^x}$

$f'(x)=\frac{(e^x)(2x)-(x^2)(e^x)}{(e^x)^2}$

$=\frac{(e^x)(2x-x^2)}{(e^x)^2}$

$=\frac{2x-x^2}{e^x}$

7 $f(x)=\frac{x^2}{x^2+1}$

$f'(x)=\frac{(x^2+1)(2x)-(x^2)(2x)}{(x^2+1)^2}$

$=\frac{2x^3+2x-2x^3}{(x^2+1)^2}$

$=\frac{2x}{(x^2+1)^2}$

8 $f(x)=3x\ln x$

$f'(x)=(3x)\left(\frac{1}{x}\right)+(\ln x)(3)$

$=3+3\ln x$

9 $f(x)=\frac{x^2-2x+1}{x}$

$=x-2+x^{-1}$

$f'(x)=1-\frac{1}{x^2}$

10 $f(x)=\sqrt{x}(x^2+1)$

$=x^{\frac{5}{2}}+x^{\frac{1}{2}}$

$f'(x)=\frac{5}{2}x^{\frac{3}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$

$=\frac{5}{2}x^{\frac{3}{2}}+\frac{1}{2x^{\frac{1}{2}}}$

11 $f(x)=\frac{x}{x^2-2x+1}$

$f'(x)=\frac{(x^2-2x+1)(1)-(x)(2x-2)}{(x^2-2x+1)^2}$

$=\frac{(x^2-2x+1)-(2x^2-2x)}{(x^2-2x+1)^2}$

$=\frac{-x^2+1}{(x^2-2x+1)^2}$

$=\frac{-(x-1)(x+1)}{(x-1)^4}$

$=-\frac{(x+1)}{(x-1)^3}$

12 $f(x)=(x^3-3x)(2x^2+3x+5)$

$f'(x)=(x^3-3x)(4x+3)+(2x^2+3x+5)(3x^2-3)$

$=(4x^4+3x^3-12x^2-9x)+$

$(6x^4+9x^3+9x^2-9x-15)$

$=10x^4+12x^3-3x^2-18x-15$

13 $f(x)=xe^x-e^x$

$f(1)=(1)(e^1)-e^1$

$=0$

$f'(x)=\left[(x)(e^x)+(e^x)(1)\right]-e^x$

$=xe^x$

$m_{\text{tangent}}=f'(1)$

$=(1)(e^1)$

$=e$

$m_{\text{normal}}=-\frac{1}{e}$

$y-0=-\frac{1}{e}(x-1)$

14 $f(x)=x^3\ln x$

$f(1)=(1^3)(\ln 1)$

$=0$

$f'(x)=(x^3)\left(\frac{1}{x}\right)+(\ln x)(3x^2)$

$=x^2+3x^2\ln x$

$m=f'(1)$

$=1^2+3(1^2)(\ln 1)$

$=1$

$y-0=1(x-1)$ or $y=x-1$

15 $c(n)=-4.5n^2+3.5n-2$

$\frac{\mathrm{d}c}{\mathrm{d}n}=9n+3.5$

16 $A=\frac{4}{3}\pi r^3$

$\frac{\mathrm{d}A}{\mathrm{d}r}=\left(\frac{4}{3}\pi\right)(3r^2)$

$=4\pi r^2$

17 $v(t)=2t^2-t+1$

$\frac{\mathrm{d}v}{\mathrm{d}t}=4t-1$

$\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=2}=4(2)-1$

$=7$

18 $\frac{\mathrm{d}}{\mathrm{d}t}\left[(e^t)(t+3)\right]=(e^t)(1)+(t+3)(e^t)$

$=e^t(t+4)$

$e^t(t+k)\Rightarrow k=4$

Exercise 7K

1 $f(x)=(3x^4+2x)^5$

$u(x)=x^5$

$v(x)=3x^4+2x$

$f'(x)=5(3x^4+2x)^4(12x^3+2)$ or

$10(3x^4+2x)^4(6x^3+1)$

2 $f(x)=4(2x^2+3x+1)^3$

$u(x)=4x^3$

$v(x)=2x^2+3x+1$

$f(x)=12(2x^2+3x+1)^2(4x+3)$

3 $f(x)=\ln(3x^5)$

$u(x)=\ln x$

$v(x)=3x^5$

$f'(x)=\left(\frac{1}{3x^5}\right)(15x^4)$

$=\frac{5}{x}$

4 $f(x) = \sqrt[3]{2x+3}$

$u(x) = x^{\frac{1}{3}}$

$v(x) = 2x+3$

$f'(x) = \left[\frac{1}{3}(2x+3)^{-\frac{2}{3}}\right](2)$

$= \frac{2}{3(2x+3)^{\frac{2}{3}}}$

5 $f(x) = e^{4x}$

$u(x) = e^x$

$v(x) = 4x$

$f'(x) = (e^{4x})(4) = 4e^{4x}$

6 $f(x) = (\ln x)^3$

$u(x) = x^3$

$v(x) = \ln x$

$f'(x) = 3(\ln x)^2\left(\frac{1}{x}\right)$

$= \frac{3(\ln x)^2}{x}$

7 $f(x) = (9x+2)^{\frac{2}{3}}$

$u(x) = x^{\frac{2}{3}}$

$v(x) = 9x+2$

$f'(x) = \left[\frac{2}{3}(9x+2)^{-\frac{1}{3}}\right](9)$

$= \frac{6}{(9x+2)^{\frac{1}{3}}}$

8 $f(x) = \sqrt[4]{2x^2+3}$

$u(x) = x^{\frac{1}{4}}$

$v(x) = 2x^2+3$

$f'(x) = \left[\frac{1}{4}(2x^2+3)^{-\frac{3}{4}}\right](4x)$

$= \frac{x}{(2x^2+3)^{\frac{3}{4}}}$

9 $f(x) = 5(x^3+3x)^4$

$u(x) = 5x^4$

$v(x) = x^3+3x$

$f'(x) = 20(x^3+3x)^3(3x^2+3)$ or

$60(x^3+3x)^3(x^2+1)$

10 $f(x) = e^{4x^3}$

$u(x) = e^x$

$v(x) = 4x^3$

$f'(x) = \left(e^{4x^3}\right)(12x^2) = 12x^2e^{4x^3}$

Exercise 7L

1 $f(x) = x^2(2x-3)^4$

$f'(x) = (x^2)\left[4(2x-3)^3(2)\right] + (2x-3)^4(2x)$

$= 8x^2(2x-3)^3 + 2x(2x-3)^4$

$= 2x(2x-3)^3(4x+2x-3)$

$= 2x(2x-3)^3(6x-3)$

$= 6x(2x-1)(2x-3)^3$

2 $f(x) = x^2e^{-x}$

$f'(x) = (x^2)\left[e^{-x}(-1)\right] + (e^{-x})(2x)$

$= e^{-x}(-x^2+2x)$

$= \frac{-x^2+2x}{e^x}$

3 $f(x) = \frac{4}{x^2+3}$

$= 4(x^2+3)^{-1}$

$f'(x) = \left[-4(x^2+3)^{-2}\right](2x)$

$= \frac{-8x}{(x^2+3)^2}$

4 $f(x) = \frac{x}{\sqrt{2x+1}}$

$= \frac{x}{(2x+1)^{\frac{1}{2}}}$

$f'(x) = \frac{(2x+1)^{\frac{1}{2}}(1) - (x)\left[\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)\right]}{\left((2x+1)^{\frac{1}{2}}\right)^2}$

$= \frac{(2x+1)^{\frac{1}{2}} - \frac{x}{(2x+1)^{\frac{1}{2}}}}{2x+1} \cdot \frac{(2x+1)^{\frac{1}{2}}}{(2x+1)^{\frac{1}{2}}}$

$= \frac{(2x+1)-x}{(2x+1)^{\frac{3}{2}}}$

$= \frac{x+1}{(2x+1)^{\frac{3}{2}}}$

5 $f(x) = \sqrt{e^{2x}+e^{-2x}}$

$= (e^{2x}+e^{-2x})^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(e^{2x}+e^{-2x})^{-\frac{1}{2}}\left[(e^{2x})(2) + (e^{-2x})(-2)\right]$

$= \frac{1}{2}(e^{2x}+e^{-2x})^{-\frac{1}{2}}\left[2(e^{2x}-e^{-2x})\right]$

$= \frac{e^{2x}-e^{-2x}}{(e^{2x}+e^{-2x})^{\frac{1}{2}}}$

6 $f(x) = \ln(1-2x^3)$

$f'(x) = \frac{1}{1-2x^3}(-6x^2)$

$= \frac{-6x^2}{1-2x^3}$ or $\frac{6x^2}{2x^3-1}$

7 $f(x) = \ln(\ln x^2)$

$f'(x) = \left(\frac{1}{\ln x^2}\right)\left[\left(\frac{1}{x^2}\right)(2x)\right]$

$= \frac{2}{x\ln x^2}$

$= \frac{2}{2x\ln x} = \frac{1}{x\ln x}$

8 $f(x) = \frac{2}{e^x + e^{-x}}$

$= 2(e^x + e^{-x})^{-1}$

$f'(x) = \left[-2(e^x + e^{-x})^{-2}\right]\left(e^x + e^{-x}(-1)\right)$

$= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$

9 $f(x) = \frac{1}{x^2 - 3x - 2}$

$= \left(x^2 - 3x - 2\right)^{-1}$

$f'(x) = \left[-1\left(x^2 - 3x - 2\right)^{-2}\right](2x - 3)$

$= \frac{-2x + 3}{\left(x^2 - 3x - 2\right)^2}$

10 $f(x) = x^4\sqrt{x^2 + 3}$

$= x^4(x^2 + 3)^{\frac{1}{2}}$

$f'(x) = (x^4)\left[\frac{1}{2}(x^2 + 3)^{-\frac{1}{2}}(2x)\right] + (x^2 + 3)^{\frac{1}{2}}(4x^3)$

$= \frac{x^5}{(x^2 + 3)^{\frac{1}{2}}} + 4x^3(x^2 + 3)^{\frac{1}{2}}$

$= \frac{x^5}{(x^2 + 3)^{\frac{1}{2}}} + 4x^3(x^2 + 3)^{\frac{1}{2}} \cdot \frac{(x^2 + 3)^{\frac{1}{2}}}{(x^2 + 3)^{\frac{1}{2}}}$

$= \frac{x^5}{(x^2 + 3)^{\frac{1}{2}}} + \frac{4x^3(x^2 + 3)}{(x^2 + 3)^{\frac{1}{2}}}$

$= \frac{5x^5 + 12x^3}{(x^2 + 3)^{\frac{1}{2}}}$

11 **a** $f(x) = e^{x^2 - 2x}$

$f'(x) = \left(e^{x^2 - 2x}\right)(2x - 2)$

b $f'(2) = \left(e^{2^2 - 2(2)}\right)(2(2) - 2)$

$= e^0(2)$

$= 2$

c $f(2) = e^{2^2 - 2(2)}$

$= 1$

The tangent at (2, 1) has gradient $m = f'(2) = 2$

$y - 1 = 2(x - 2)$

12 $f(x) = x^3 \ln x$

$f'(x) = (x^3)\left(\frac{1}{x}\right) + (\ln x)(3x^2)$

$= x^2 + 3x^2 \ln x$

horizontal tangent line $\Rightarrow f'(x) = 0$.

$x^2 + 3x^2 \ln x = 0$

$x^2(1 + 3\ln x) = 0$

$x^2 = 0$ or $1 + 3\ln x = 0$

$x = 0$ or $\ln x = -\frac{1}{3}$

$e^{\ln x} = e^{-\frac{1}{3}}$

$x = 0$ or $x = \frac{1}{\sqrt[3]{e}}$

13 $h(x) = (f \circ g)(x)$

$= f(1 - 2x)$

$= \frac{1}{(1 - 2x)^3}$

The gradient of h is $h'(x)$.

$h(x) = \frac{1}{(1 - 2x)^3}$

$= (1 - 2x)^{-3}$

$h'(x) = \left[-3(1 - 2x)^{-4}\right](-2)$

$= \frac{6}{(1 - 2x)^4}$

Since $6 < 0$ and $(1 - 2x)^4 > 0$ for all x where h is defined, the gradient of h is always positive.

14

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	4	−3	2
4	2	−1	3	4

a $(f \circ g)'(x) = \left[f'(g(x))\right]g'(x)$

$(f \circ g)'(3) = \left[f'(g(3))\right]g'(3)$

$= \left[f'(4)\right]g'(3)$

$= 3(2)$

$= 6$

b $\frac{d}{dx}\left[\frac{1}{[g(x)]^2}\right] = \frac{d}{dx}\left[\left(g(x)\right)^{-2}\right]$

$= \left[-2\left(g(x)\right)^{-3}\right]\left(g'(x)\right)$

$= -\frac{2\left(g'(x)\right)}{\left(g(x)\right)^{-3}}$

$\frac{d}{dx}\left[\frac{1}{[g(x)]^2}\right]_{x=4} = -\frac{2\left(g'(4)\right)}{\left(g(4)\right)^{-3}} = -\frac{2(4)}{(-1)^3}$

$= 8$

Exercise 7M

1 $f(x) = 4x^{\frac{3}{2}}$

$f'(x) = 4\left(\frac{3}{2}x^{\frac{1}{2}}\right)$

$= 6x^{\frac{1}{2}}$

$f''(x) = 6\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \frac{3}{\sqrt{x}}$

2 $f(x) = 3x^5 + x^4 + 2x + 1$

$f''(x) = 15x^4 + 4x^3 + 2$

$f''(x) = 60x^3 + 12x^2$

$f'''(x) = 180x^2 + 24x$

3 $C(n) = (3 + 2n)e^{-3n}$

$\frac{dC}{dn} = (3 + 2n)\left[\left(e^{-3n}\right)(-3)\right] + \left(e^{-3n}\right)(2)$

$= \left(e^{-3n}\right)\left[(3 + 2n)(-3) + 2\right]$

$= \left(e^{-3n}\right)(-7 - 6n)$

$\frac{d^2C}{dn^2} = \left(e^{-3n}\right)(-6) + (-7 - 6n)\left[\left(e^{-3n}\right)(-3)\right]$

$= 3e^{-3n}\left[-2 - (-7 - 6n)\right]$

$= 3e^{-3n}(6n + 5)$

4 $\frac{dy}{dx} = \frac{4}{x}$

$= 4x^{-1}$

$\frac{d^2y}{dx^2} = -4x^{-2}$

$\frac{d^3y}{dx^3} = 8x^{-3}$

$= \frac{8}{x^3}$

5 $\frac{d^4y}{dx^4} = \ln(4x^3)$

$\frac{d^5y}{dx^5} = \left(\frac{1}{4x^3}\right)(12x^2)$

$= \frac{3}{x}$ or $3x^{-1}$

$\frac{d^6y}{dx^6} = -3x^{-2}$

$= \frac{-3}{x^2}$

6 $R(t) = \frac{1}{2}t\ln(t^2)$

$\frac{dR}{dt} = \left(\frac{1}{2}t\right)\left[\left(\frac{1}{t^2}\right)(2t)\right] + \left(\ln(t^2)\right)\left(\frac{1}{2}\right)$

$= 1 + \frac{1}{2}\ln(t^2)$

$\left.\frac{dR}{dt}\right|_{t=-1} = 1 + \frac{1}{2}\ln((-1)^2)$

$= 1 + \frac{1}{2}(0)$

$= 1$

7 $y = x^3 + 3x^2 + 2x + 4$

$y'(x) = 3x^2 + 6x + 2$

$y''(x) = 6x + 6$

$y'''(x) = 6$

$y^{(4)}(x) = 0$

$y^{(5)}(x) = 0$

So for $n \geq 4$, $\frac{d^ny}{dx^n}$ equals 0

8 $y = e^x + e^{-x}$

$\frac{dy}{dx} = e^x - e^{-x}$

$\frac{d^2y}{dx^2} = e^x - e^{-x}(-1) = e^x + e^{-x}$

$\frac{d^3y}{dx^3} = e^x + e^{-x}(-1) = e^x - e^{-x}$

$\frac{d^4y}{dx^4} = e^x - e^{-x}(-1) = e^x + e^{-x}$

When n is odd $\frac{d^ny}{dx^n} = e^x - e^{-x}$ and when n is even $\frac{d^ny}{dx^n} = e^x + e^{-x}$

9 $y = \frac{1}{x} = x^{-1}$

$\frac{dy}{dx} = -x^{-2} = \frac{-1}{x^2}$

$\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$

$\frac{d^3y}{dx^3} = -6x^{-4} = \frac{-6}{x^4}$

$\frac{d^4y}{dx^4} = 24x^{-5} = \frac{24}{x^5}$

So $\frac{d^ny}{dx^n} = \frac{(-1)^n n!}{x^{n+1}}$

10 The slope of $f(x) = 3\sqrt[5]{x^2}$ is $f'(x)$ and so the gradient of the slope is $f''(x)$.

$f(x) = 3\sqrt[5]{x^2}$

$= 3x^{\frac{2}{5}}$

$f'(x) = \frac{6}{5}x^{-\frac{3}{5}}$

$f''(x) = -\frac{18}{25}x^{-\frac{8}{5}}$

$= \frac{-18}{25x^{\frac{8}{5}}}$

Exercise 7N

1 a $h(0) = -4.9(0^2) + 19.6(0) + 1.4 = 1.4$ m

$h(2) = -4.9(2^2) + 19.6(2) + 1.4 = 21$ m

b average speed $= \frac{\text{distance travelled}}{\text{time taken}}$

$= \frac{h(2) - h(0)}{2 - 0} = \frac{19.6 \text{ m}}{2 \text{ s}}$

$= 9.8 \text{ ms}^{-1}$

c $v(t) = h'(t)$
$= -9.8t + 19.6$
$v(1) = 9.8\ \text{ms}^{-1}$
$v(2) = 0\ \text{ms}^{-1}$
$v(3) = -9.8\ \text{ms}^{-1}$

The ball is moving upward at 1 s, at rest at 2 s and downward at 3 s.

2 **a** $V(t) = 4000\left(1 - \frac{t}{60}\right)^2$

$V(0) = 4000$ gallons

V(20)=1778 gallons

b $\frac{V(2) - V(0)}{20 - 0} = \frac{-2222 \text{ gal}}{20 \text{ min}} \approx -111 \frac{\text{gal}}{\text{min}}$

During the time interval 0 to 20 minutes, water is being drained from the tank at an average rate of 111 gallons per minute.

c $V(t) = 4000\left(1 - \frac{t}{60}\right)^2$
$V'(t) = 8000\left(1 - \frac{t}{60}\right)\left(\frac{-1}{60}\right)$
$= \frac{-400}{3}\left(1 - \frac{t}{60}\right)$
$V'(20) = -89 \frac{\text{gal}}{\text{min}}$

At 20 minutes, water is being drained from the tank at an average rate of 89 gallons per minute.

d $V'(t) = \frac{-400}{3}\left(1 - \frac{t}{60}\right)$

$\frac{-400}{3}\left(1 - \frac{t}{60}\right) = 0$

$1 - \frac{t}{60} = 0$

$t = 60$

$V'(t)$ is negative for $0 \le t < 60$ minutes, which means water is flowing out of the tank during this time interval. Therefore the amount of water in the tank is never increasing from $t = 0$ minutes to $t = 40$ minutes.

3 **a** $P(t) = 100e^{0.25t}$
$\frac{P(10) - P(0)}{10 - 0} \approx \frac{1118 \text{ bacteria}}{10 \text{ day}} = 112 \frac{\text{bacteria}}{\text{day}}$

b $P(t) = 100e^{0.25t}$
$P'(t) = 100e^{0.25t}(0.25)$
$P'(t) = 25e^{0.25t}$

c $P'(10) = 25e^{0.25t}$
$\approx 305 \frac{\text{bacteria}}{\text{day}}$

End of day 10 the number of bacteria are increasing at rate of 305 bacteria/day.

4 **a** $C(n) = 0.05n^2 + 10n + 5000$

$\frac{C(105) - C(100)}{105 - 100} = \frac{101.25 \text{ dollars}}{5 \text{ units}}$
$= 20.25 \frac{\text{dollars}}{\text{unit}}$

$\frac{C(101) - C(100)}{101 - 100} = \frac{20.05 \text{ dollars}}{1 \text{ unit}}$
$= 20.05 \frac{\text{dollars}}{\text{unit}}$

b $C'(n) = 0.1n + 10$

c $C'(100) = 0.1(100) + 10$
$= 20 \frac{\text{dollars}}{\text{unit}}$

It costs more than 20 dollars per unit to produce units after the 100th unit.

Exercise 7O

1 **a** $s(t) = t^3 - 6t^2 + 9t$
$v(t) = s'(t)$
$= 3t^2 - 12t + 9$
$s(0) = 0\,\text{cm}$
$v(0) = 9\ \text{cm s}^{-1}$

b The particle is at rest when $v(t) = 0$

$3t^2 - 12t + 9 = 0$
$3(t^2 - 4t + 3) = 0$
$3(t - 1)(t - 3) = 0$
$t = 1\text{s}, 3\text{s}$

c Use the initial displacement and find the displacement of the particle when it is at rest.

$s(1) = 4$ cm; $s(3) = 0$ cm

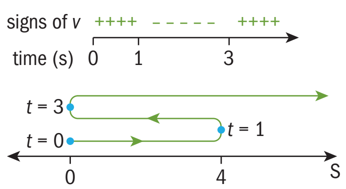
Make a sign diagram for velocity to see which direction the particle is moving.

Choose a value of t in each interval and find the sign of $v(t)$:

$t \in (0, 1)$: $t = \frac{1}{2}$, $v\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) + 9$
$= 3.75 > 0$

$t \in (1, 3)$: $t = 2$, $v(2) = 3(2)^2 - 12(2) + 9$
$= -3 < 0$

$t \in (3, \infty)$: $t = 4$, $v(4) = 3(4)^2 - 12(4) + 9 = 9 > 0$



2 **a** $s(t) = -16t^2 + 40t + 4$
$s(0) = 4\,\text{ft}$

b $s(2) = -16(2^2) + 40(2) + 4$
$= -64 + 80 + 4 = 20\,\text{ft}$

c **i** $-16t^2+40t+4=20$

ii $-16t^2+40t+4=20$

$-16t^2+40t-16=0$

$-8(2t^2-5t+2)=0$

$-2(2t-1)(t-2)=0$

$t=\frac{1}{2}\text{ s}, 2\text{ s}$

d **i** $s(t)=-16t^2+40t+4$

$\frac{ds}{dt}=-32t+40$

ii $v(t)=-32t+40$

$v(0)=40 \text{ ft s}^{-1}$

iii $-32t+40=0$

$t=\frac{5}{4}\text{ s}$

iv The maximum occurs when velocity equals 0 ie when $t=\frac{5}{4}$ s.

$s\left(\frac{5}{4}\right)=-16\left(\frac{5}{4}\right)^2+40\left(\frac{5}{4}\right)+4$

$= -25 + 50 + 4$

$= 29$ ft

3 **a** $v=\frac{ds}{dt}=-te^{-t}+e^{-t}$

$=e^{-t}(1-t)$

$=\frac{1-s}{e^t}$

b $v=0$ when $t=1$

So particle is at rest after 1 second

Exercise 7P

1 **a** $s(t)=2t^4-6t^2$

$v(t)=s'(t)$

$v(t)=8t^3-12t,\ t\geq 0$

$a(t)=v'(t)$

$a(t)=24t^2-12,\ t\geq 0$

b $a(t)=24t^2-12$

$a(2)=84 \text{ cm s}^{-2}$

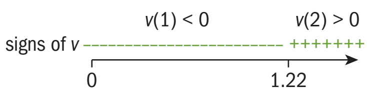
Velocity is increasing 84 cm s^{-2} at time 2 seconds.

c $v(t)=0$

$8t^3-12t=0$

$4t(2t^2-3)=0$

$t=0\text{ s},\ 1.22\text{ s}$

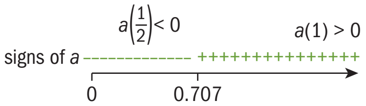


$a(t)=0$

$24t^2-12=0$

$12(2t^2-1)=0$

$t=0.707$ s



Speeding up when velocity and acceleration have the same sign: $0<t<0.707$ and $t>1.22$ s. Slowing down when velocity and acceleration have the different signs: $0.707<t<1.22$s

2 **a** $s(t)=-t^3+12t^2-36t+20$

$v(t)=s'(t)$

$v(t)=-3t^2+24t-36$

$a(t)=v'(t)$

$a(t)=-6t+24$

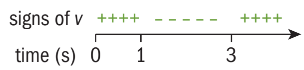
v(1)<0 v(3)>0 v(7)<0

b $s(0)=20$ ft

$v(0)=-36$ ft/s

$a(0)=24$ ft/s^2

c $v(t)=0$



$-3t^2+24t-36=0$

$-3(t^2-8t+12)=0$

$-3(t-2)(t-6)=0$

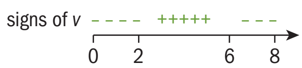
$t=2\text{ s}, 6\text{ s}$

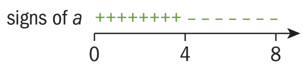
The particle is moving left on (0, 2) and (6, 8), and moving right on (2, 6).

d $a(t)=0$

$-6t+24=0$

$t=4$ s





Speeding up when velocity and acceleration have the same sign: (2, 4) and (6, 8)

Slowing down when velocity and acceleration have the different signs: (0, 2) and (4, 6)

3 **a** $s(t)=-4.9t^2+4.9t+10$

$v(t)=s'(t)$

$v(t)=-9.8t+4.9$

$a(t)=v'(t)$

$a(t)=-9.8$

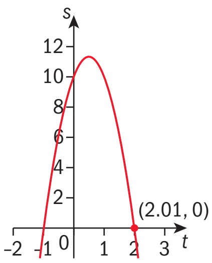
b The diver hits the water when $s(t) = 0$

$s(t)=0$

$-4.9t^2+4.9t+10=0$

$t=2.01$s

Use a GDC to solve for the value of t greater than 0.



c The maximum height occurs when velocity is equal to 0.

$v(t) = 0$

$-9.8t + 4.9 = 0$

$t = 0.5$

$s(0.5) = 11.225\,\text{m}$

d $v(0.3) = 1.96 > 0$

$a(0.3) = -9.8 < 0$

Since the signs of $v(0.3)$ and $a(0.3)$ are different the particle is slowing down at 0.3 seconds.

4 a $s(t) = \frac{1}{4}t^2 - \ln(t+1)$

$v(t) = \frac{ds}{dt} = \frac{1}{2}t - \frac{1}{t+1}$

particle at rest when $v(t) = 0$

$\frac{1}{2}t(t+1) - 1 = 0$

$\frac{1}{2}t^2 + \frac{1}{2}t - 1 = 0$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

when $t = 1$

b $a(t) = \frac{dv}{dt} = \frac{1}{2}t + \frac{1}{(t+1)^2}$

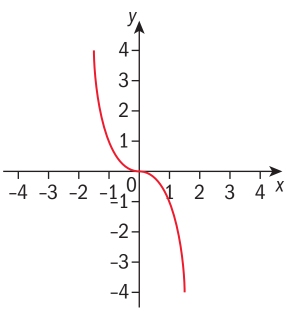
as $t > 0$ $a(t) > 0$

therefore velocity is never decreasing.

Exercise 7Q

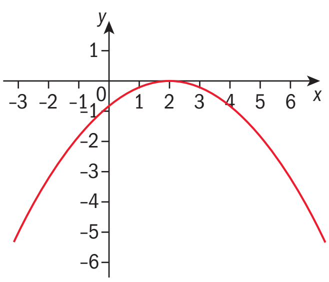
In questions 1-3, if y increases as x increases the function is increasing. If y decreases as x increases the function is decreasing.

1



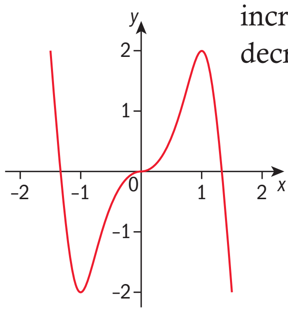
decreasing $(-\infty, \infty)$

2



increasing $(-\infty, 2)$
decreasing $(2, \infty)$

3



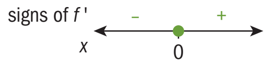
increasing $(-1, 1)$
decreasing $(-\infty, -1)$ and $(1, \infty)$

4 $f(x) = x^4$

$f'(x) = 4x^3$

$4x^3 = 0$

$x = 0$



increasing when $f'(x) > 0$: $(0, \infty)$
decreasing when $f'(x) < 0$: $(-\infty, 0)$

5 $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x$

$4x(x^2 - 1) = 0$

$4x(x+1)(x-1) = 0$

$x = 0, -1, 1$

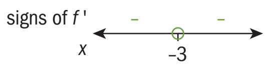
increasing when $f'(x) > 0$: $(-1, 0)$ and $(1, \infty)$
decreasing when $f'(x) < 0$: $(-\infty, -1)$ and $(0, 1)$

6 $f(x) = \frac{x+2}{x-3}$

$f'(x) = \frac{(x-3)(1) - (x+2)(1)}{(x-3)^2}$

$= \frac{(x-3)(1) - (x+2)(1)}{(x-3)^2}$

$= \frac{-5}{(x-3)^2}$



$f'(x) \neq 0$

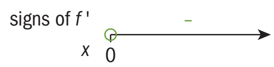
$f'(x)$ undefined when $x = 3$

decreasing when $f'(x) < 0$: $(-\infty, 3)$ and $(3, \infty)$

7 $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{-1}{2x^{\frac{3}{2}}}$

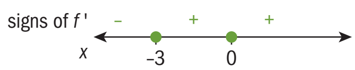


$f'(x) \neq 0$

domain $x > 0$

decreasing when $f'(x) > 0$: $(0, \infty)$

8 $f(x) = x^3e^x$



$f'(x) = (x^3)(e^x) + (e^x)(3x^2)$

$e^x x^2(x+3) = 0$

$x = 0, -3$

increasing when $f'(x) > 0$: $(-3, \infty)$
decreasing when $f'(x) < 0$: $(-\infty, -3)$

9 $f(x)=\frac{x^3}{x^2-1}$

$f'(x)=\frac{(x^2-1)(3x^2)-(x^3)(2x)}{(x^2-1)^2}$

$=\frac{x^4-3x^2}{(x^2-1)^2}$

$=\frac{x^2(x^2-3)}{(x^2-1)^2}$

$f'(x)$ undefined when $(x^2-1)^2=0$ or $x=-1,1$.

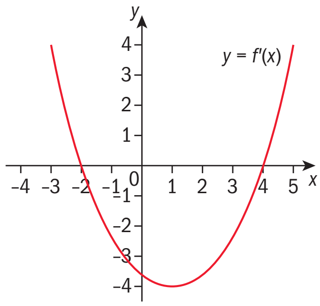
$f'(x)=0$ when $x^2(x^2-3)=0$ or $x=0,-\sqrt{3},\sqrt{3}$.

increasing when $f'(x)>0$: $(-\infty,-\sqrt{3})$ and $(\sqrt{3},\infty)$

decreasing when $f'(x)<0$: $(-\sqrt{3},-1)$ $(-1,1)$ $(1,\sqrt{3})$

10 f is increasing when $f'(x)>0$: $(-\infty,-2)$ and $(4,\infty)$

f is decreasing when $f'(x)<0$: $(-2,4)$

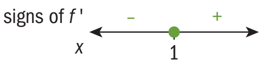


Exercise 7R

1 $f(x)=2x^2-4x-3$

$f'(x)=4x-4$

$4x-4=0$



$4(x-1)=0$

$x=1$

$f(1)=-5$

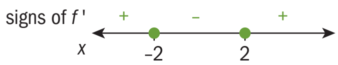
Since $f'(x)$ changes from negative to positive at $x=1$ there is a relative minimum at $x=1$

The relative minimum point is $(1,-5)$

2 $f(x)=x^3-12x-5$

$f'(x)=3x^2-12$

$3x^2-12=0$



$3(x+2)(x-2)=0$

$x=-2,2$

$f(-2)=11$

$f(2)=-21$

Since $f'(x)$ changes from negative to positive at $x=2$ there is a relative minimum at $x=2$

The relative minimum point is $(2,-21)$

Since $f'(x)$ changes from positive to negative at $x=-2$ there is a relative maximum at $x=-2$. The relative maximum point is $(-2,11)$

3 $f(x)=x^{\frac{5}{3}}$

$f'(x)=\frac{5}{3}x^{\frac{2}{3}}$

$\frac{5}{3}x^{\frac{2}{3}}=0$

$x=0$

for $x<0, f'(x)>0$

for $x>0, f'(x)>0$

Since $f'(x)$ does not change signs there are no relative maximum or minimum points.

4 $f(x)=x^4-2x^2$

$f'(x)=4x^3-4x$

$4x^3-4x=0$

$4x(x+1)(x-1)=0$

$x=-1,0,1$

$f(0)=0$

$f(-1)=-1$

$f(1)=-1$

Since $f'(x)$ changes from negative to positive at $x=-1$ and $x=1$ there are relative minimums at $x=-1$ and $x=1$

The relative minimum points are $(-1,-1)$ and $(1,-1)$

Since $f'(x)$ changes from positive to negative at $x=0$ there is a relative maximum at $x=0$.

The relative maximum point is $(0,0)$

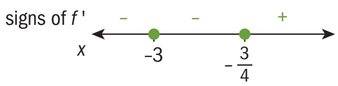
5 $f(x)=x(x+3)^3$

$f'(x)=(x)\left[3(x+3)^2(1)\right]+(x+3)^3(1)$

$=(x+3)^2(4x+3)$

$(x+3)^2(4x+3)=0$

$x=-3,-\frac{3}{4}$



$f\left(-\frac{3}{4}\right)=-\frac{2187}{256}$

Since $f'(x)$ changes from negative to positive at $x=-\frac{3}{4}$ there is a relative minimum at $x=-\frac{3}{4}$

The relative minimum point is $\left(-\frac{3}{4},-\frac{2187}{256}\right)$

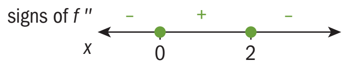
There is no relative extremum at $x=-3$ since $f'(x)$ does not change signs at $x=-3$

6 $f(x)=x^2e^{-x}$

$f'(x)=(x^2)\left[(e^{-x})(-1)\right]+(e^{-x})(2x)$

$=xe^{-x}(-x+2)$

$xe^{-x}(-x+2)=0$



$x=0,2$

$f(0)=0$

$f(2)=\frac{4}{e^2}$

Since $f'(x)$ changes from negative to positive at $x = 0$ there is a relative minimum at $x = 0$

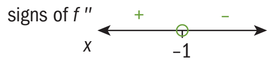
The relative minimum point is (0, 0)

Since $f'(x)$ changes from positive to negative at $x = 2$ there is a relative maximum at $x = 2$.

The relative maximum point is $\left(2, \frac{4}{e^2}\right)$

7 $f(x)=\frac{1}{(x+1)^2}=(x+1)^{-2}$

$f'(x)=-2(x+1)^{-3}(1)$



$=\frac{-2}{(x+1)^3}$

$f'(x) \neq 0$

$f'(x)$ undefined when $(x+1)^3=0$ or $x=-1$

Although f changes signs at $x = -1$, there is no relative extremum since f is undefined at $x = -1$

8 $f(x)=\frac{x^2-2x+1}{x+1}$

$f'(x)=\frac{(x+1)(2x-2)-(x^2-2x+1)(1)}{(x+1)^2}$

$=\frac{(2x^2-2)-(x^2-2x+1)}{(x+1)^2}$

$=\frac{x^2+2x-3}{(x+1)^2}$

$=\frac{(x+3)(x-1)}{(x+1)^2}$

$f'(x)$ undefined when $(x+1)^2=0$ or $x=-1$

$f'(x) = 0$ when $(x+3)(x-1)=0$ or $x=-3,1$

$f'(-3) = -8$

$f(1) = 0$

Since $f'(x)$ changes from negative to positive at $x = 1$ there is a relative minimum at $x = 1$

The relative minimum point is (1, 0)

Since $f'(x)$ changes from positive to negative at $x = -3$ there is a relative maximum at $x = -3$.

The relative maximum point is (−3, −8)

Exercise 7S

1 $f(x)=2x^2-4x-3$

$f'(x)=4x-4$

$f''(x)=4$

Since $f''(x) > 0$ for all x, f is concave up on $(-\infty, \infty)$

There are no inflexion points.

2 $f(x)=-x^4+4x^3$

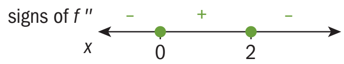
$f'(x)=-4x^3+12x^2$

$f''(x)=-12x^2+24x$

$f''(x)=0$ for inflexion points

$-12x^2+24x=0$

$-12x(x-2)=0$



$x=0,2$

$f(0)=0$

$f(2)=16$

concave up when $f''(x) > 0$: (0, 2)

concave down when $f''(x) < 0$: $(-\infty, 0)$ and $(2, \infty)$ inflexion points: (0, 0) and (2, 16)

3 $f(x)=x^3-6x^2+12x$

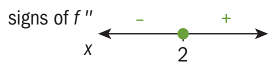
$f'(x)=3x^2-12x+12$

$f''(x)=6x-12$

$f''(x)=0$ for inflexion points

$6x-12=0$

$6(x-2)=0$



$x=2$

$f(2)=8$

concave up when $f''(x) > 0$: $(2, \infty)$

concave down when $f''(x) < 0$: $(-\infty, 2)$

inflexion point: (2, 8)

4 $f(x)=x^4$

$f'(x)=4x^3$

$f''(x)=12x^2$

$f''(x)=0$ for inflexion points

$12x^2=0$

$x=0$

concave up when $f''(x) > 0$: $(-\infty, \infty)$

There are no inflexion points since $f''(x)$ does not change sign either side of $x = 0$.

5 $f(x)=2xe^x$

$f'(x)=(2x)(e^x)+(e^x)(2)$

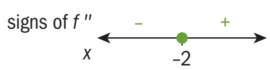
$=2e^x(x+1)$

$f''(x)=(2e^x)(1)+(x+1)(2e^x)$

$=2e^x(x+2)$

$f''(x)=0$ for inflexion points

$2e^x(x+2)=0$



$x=-2$

$f(-2)=-\frac{4}{e^2}$

concave up when $f''(x) > 0$: $(-2, \infty)$

concave down when $f''(x) < 0$: $(-\infty, -2)$

inflexion point: $\left(-2, -\frac{4}{e^2}\right)$

6 $f(x)=\frac{1}{x^2+1}=(x^2+1)^{-1}$

$f'(x)=-(x^2+1)^{-2}(2x)$

$=\frac{-2x}{(x^2+1)^2}$

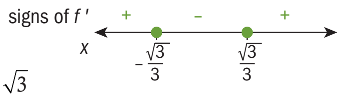
$f''(x)=\frac{(x^2+1)^2(-2)-(-2x)\left[2(x^2+1)(2x)\right]}{\left[(x^2+1)^2\right]^2}$

$=\frac{-2(x^2+1)\left[(x^2+1)-4x^2\right]}{(x^2+1)^4}$

$=\frac{-2(-3x^2+1)}{(x^2+1)^3}$

$f''(x)=0$ for inflexion points

$-2(-3x^2+1)=0$



$x=\pm\sqrt{\frac{1}{3}}$ or $\pm\frac{\sqrt{3}}{3}$

$f\left(-\frac{\sqrt{3}}{3}\right)=\frac{3}{4}$

$f\left(\frac{\sqrt{3}}{3}\right)=\frac{3}{4}$

concave up when $f''(x)>0$: $\left(-\infty,-\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3},\infty\right)$

concave down when $f''(x)<0$: $\left(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$

inflexion points: $\left(-\frac{\sqrt{3}}{3},\frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{3},\frac{3}{4}\right)$

7 **a** $f'(x)=\frac{-48x}{(x^2+12)^2}$

$f''(x)=\frac{(x^2+12)^2(-48)-(-48x)[2(x^2+12)(2x)]}{\left[(x^2+12)^2\right]^2}$

$=\frac{(x^2+12)^2(-48)+192x^2(x^2+12)}{(x^2+12)^4}$

$=\frac{48(x^2+12)[-(x^2+12)+4x^2]}{(x^2+12)^4}$

$=\frac{48(x^2+12)(3x^2-12)}{(x^2+12)^4}$

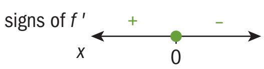
$=\frac{144(x^2+12)(x^2-4)}{(x^2+12)^4}$

$=\frac{144(x^2-4)}{(x^2+12)^3}$

b **i** $f'(x)=\frac{-48x}{(x^2+12)^2}$

$f'(x)=0$ for relative extrema

$-48x=0$



$x=0$

$f(x)=\frac{24}{x^2+12}$

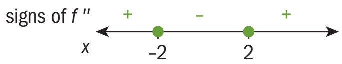
$f(0)=2$

Since $f'(x)$ changes from positive to negative at $x=0$ there is a relative maximum at $x=0$. The relative maximum point is $(0, 2)$.

ii $f''(x)=\frac{144(x^2-4)}{(x^2+12)^3}$

$f''(x)=0$ when $144(x^2-4)=0$.

$x=-2, 2$



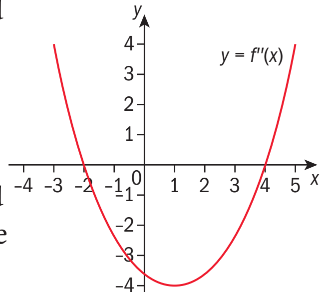
$f(x)=\frac{24}{x^2+12}$

$f(-2)=\frac{3}{2}$

$f(2)=\frac{3}{2}$

Since $f''(x)$ changes signs at $x=-2, 2$ the inflexion points are $\left(-2,\frac{3}{2}\right)$ and $\left(2,\frac{3}{2}\right)$.

8 The graph of the second derivative of f is positive for $x<-2$ and $x>4$, so f is concave up on $(-\infty,-2)$ and $(4,\infty)$. The graph of the second derivative of f is negative for $-2<x<4$, so f is concave down on $(-2, 4)$.



The inflexion points occur at $x=-2, 4$ since $f''(x)$ changes sign at $x=-2, 4$.

Exercise 7T

1 $f(x)=3x^2+10x-8$

$f(0)=-8\Rightarrow$ the y-intercept is $(0,-8)$.

$f(x)=0\ \Rightarrow 3x^2+10x-8=0$

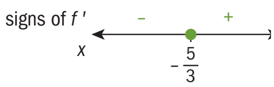
$\Rightarrow(3x-2)(x+4)=0\Rightarrow x=\frac{2}{3},-4$

The x-intercepts are $\left(\frac{2}{3},0\right)$ and $(-4, 0)$.

$f'(x)=6x+10$

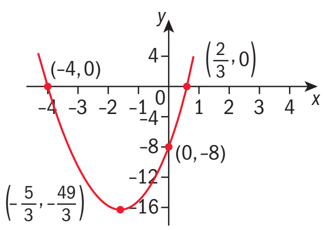
$f'(x)=0\Rightarrow 6x+10=0\Rightarrow x=-\frac{5}{3}$

$f\left(-\frac{5}{3}\right)=-\frac{49}{3}$



f has a relative minimum point at $\left(-\frac{5}{3},-\frac{49}{3}\right)$.

$f''(x)=6>0\Rightarrow f$ is always concave up.



2 $f(x)=x^3+x^2-5x-5$

$f(0)=-5 \Rightarrow$ the y-intercept is $(0,-5)$.

$f(x)=0 \ \Rightarrow x^3+x^2-5x-5=0 \Rightarrow$

$x^2(x+1)-5(x+1)=0 \Rightarrow$

$(x+1)(x^2-5)=0 \Rightarrow x=-1, \pm\sqrt{5} \Rightarrow$

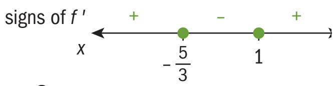
the x-intercepts are $(-1,0)$, $\left(-\sqrt{5},0\right)$

and $\left(-\sqrt{5},0\right)$

$f'(x)=3x^2+2x-5$

$f'(x)=0 \Rightarrow 3x^2+2x-5=0 \Rightarrow (3x+5)(x-1)=0 \Rightarrow$

$x=-\frac{5}{3}, 1$

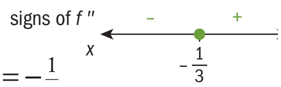


$f\left(-\frac{5}{3}\right)=\frac{40}{27}$ and $f(1)=-8$

f has a relative maximum point at $\left(-\frac{5}{3}, \frac{40}{27}\right)$

and a relative minimum point at $(1,-8)$.

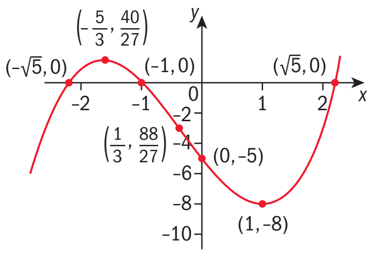
$f''(x)=6x+2$



$f''(x)=0 \Rightarrow 6x+2=0 \Rightarrow x=-\frac{1}{3}$

$f\left(-\frac{1}{3}\right)=-\frac{88}{27} \Rightarrow f$ has an inflexion point

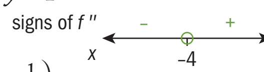
at $\left(-\frac{1}{3}, -\frac{88}{27}\right)$



3 $f(x)=\frac{x+2}{x-4}$

$f(x)$ is undefined when $x-4=0 \Rightarrow$ vertical asymptote at $x=4$

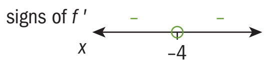
Horizontal asymptote at $y=\frac{1}{1}$ or $y=1$



$f(0)=-\frac{1}{2} \Rightarrow$ the y-intercept is $\left(0,-\frac{1}{2}\right)$

$f(x)=0$ when $x+2=0 \Rightarrow x=-2 \Rightarrow$

the $x-$intercept is $(-2, 0)$



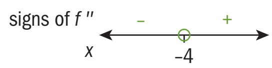
$f'(x)=\frac{(x-4)(1)-(x+2)(1)}{(x-4)^2}=\frac{-6}{(x-4)^2}$

$f'(x)$ is undefined when $(x-4)^2=0$ or when $x=4$

f' is decreasing on $(-\infty, 4)$ and $(4, \infty)$

There are no relative extrema.

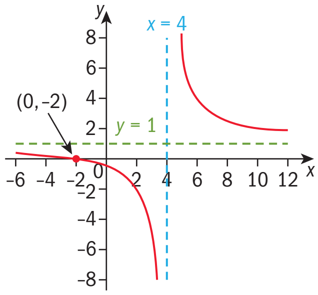
$f''(x)=12(x-4)^{-3}(1)=\frac{12}{(x-4)^3} \neq 0$



$f''(x)$ is undefined $(x-4)^3=0 \ \Rightarrow \ x=4$

f is concave down on $(-\infty, 4)$ and concave up on $(4, \infty)$

There are no inflection points.



4 $f(x)=(3-x)^4$

$f(0)=81 \Rightarrow$ the y-intercept is $(0, 81)$.

$f(x)=0 \ \Rightarrow (3-x)^4=0 \Rightarrow$

$x=3 \Rightarrow x$-intercept is $(3,0)$.

$f'(x)=4(3-x)^3(-1)=-4(3-x)^3$

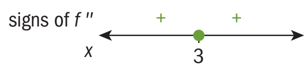
$f'(x)=0 \Rightarrow -4(3-x)^3=0 \Rightarrow x=3$

f has a relative minimum point at $(3,0)$.

$f''(x)=-12(3-x)^2(-1)=12(3-x)^2$

$f''(x)=0 \Rightarrow 12(3-x)^2=0 \Rightarrow$

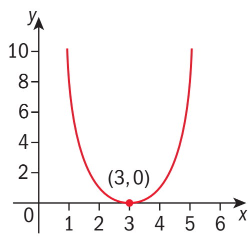
$x=3$



f'' does not change signs

at $x=3$,

so f does not have an inflection point.



5 $f(x)=\frac{e^x-e^{-x}}{2}=\frac{1}{2}(e^x-e^{-x})$

$f(0)=0 \Rightarrow$ the y-intercept is $(0, 0)$

$f(x)=0 \ \Rightarrow \frac{1}{2}(e^x-e^{-x})=0 \Rightarrow x=0 \Rightarrow$

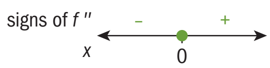
x-intercept is $(0,0)$

$f'(x)=\frac{1}{2}(e^x-e^{-x}(-1))=\frac{1}{2}(e^x+e^{-x})$

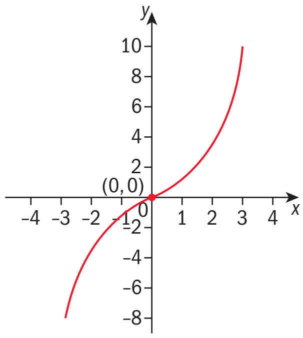
$f'(x)\neq 0$ for any $x \Rightarrow f$ has no relative extrema.

$f''(x)=\frac{1}{2}(e^x+e^{-x}(-1))=\frac{1}{2}(e^x-e^{-x})$

$f''(x)=0 \Rightarrow \frac{1}{2}(e^x-e^{-x})=0 \Rightarrow x=0$



f has an inflection point at $(0, 0)$



6 $f(x)=\frac{x^2-1}{x^2+1}$

Horizontal asymptote at

$y=\frac{1}{1}$ or $y=1$

$f(0)=-1 \Rightarrow$ the y-intercept is $(0,-1)$

$f(x)=0$ when $x^2-1=0 \Rightarrow x=-1, 1 \Rightarrow$

the x-intercepts are $(-1, 0)$ and $(1, 0)$

$f'(x)=\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$

$f'(x)=0 \Rightarrow 4x=0 \Rightarrow x=0$

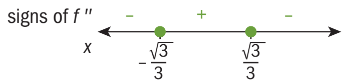
f has a relative minimum point at $(0,-1)$

$f''(x)=\frac{(x^2+1)^2(4)-(4x)\left[2(x^2+1)(2x)\right]}{\left[(x^2+1)^2\right]^2}$

$=\frac{4(x^2+1)\left[(x^2+1)-4x^2\right]}{(x^2+1)^4}=\frac{4(1-3x^2)}{(x^2+1)^3}$

$f''(x)=0 \Rightarrow 4(1-3x^2)=0 \Rightarrow$

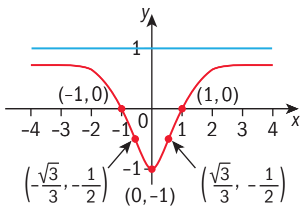
$x=\pm\sqrt{\frac{1}{3}}$ or $\pm\frac{\sqrt{3}}{3}$



$f\left(-\frac{\sqrt{3}}{3}\right)=-\frac{1}{2}$ and $f\left(\frac{\sqrt{3}}{3}\right)=-\frac{1}{2}$

f has inflection points at $\left(-\frac{\sqrt{3}}{3},-\frac{1}{2}\right)$

and $\left(\frac{\sqrt{3}}{3},\frac{1}{2}\right)$.

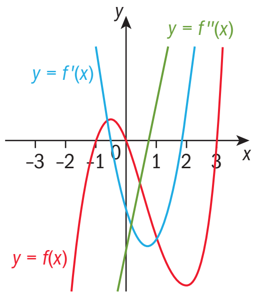


Exercise 7U

1 When $y=f(x)$ has a relative minimum or maximum, $f'(x)=0$ and so $y=f'(x)$ has an x-intercepts. When $y=f(x)$ is increasing the graph of $y=f'(x)$ is positive and when $y=f(x)$ is decreasing the graph of $y=f'(x)$ is negative.

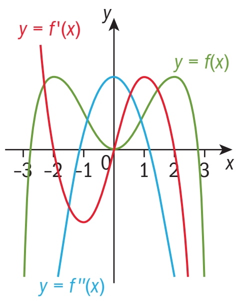
When $y=f'(x)$ has a relative minimum, $f''(x)=0$ and so $y=f''(x)$ as an x-intercept.

When $y=f'(x)$ is increasing and $y=f(x)$ is concave up the graph of $y=f''(x)$ is positive and when $y=f'(x)$ is decreasing and $y=f(x)$ is concave down the graph of $y=f''(x)$ is negative.



2 When the graph of $y=f'(x)$ has a zero and changes from positive to negative the graph of $y=f(x)$ has relative maximums and when the graph of $y=f'(x)$ has a zero and changes from negative to positive the graph of $y=f(x)$ has a relative minimum.

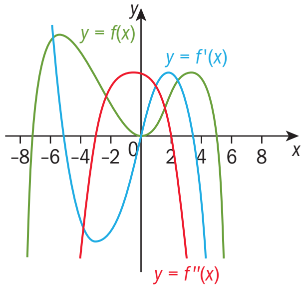
When the graph of $y=f'(x)$ has a relative maximum or minimum the graph of $y=f''(x)$ has zeros. The graph of $y=f''(x)$ is positive when the graph of $y=f'(x)$ is increasing and the graph of $y=f(x)$is concave up. The graph of $y=f''(x)$ is negative when the graph of $y=f'(x)$ is decreasing and the graph of $y=f(x)$ is concave down.



3 When the graph of $y=f''(x)$ has a zero and changes from negative to positive the graph of $y=f'(x)$ has a relative minimum and when the graph of $y=f''(x)$ has a zero and changes from positive to negative the graph of $y=f'(x)$ has a relative maximum.

When the graph of $y=f'(x)$ has a zero and changes from positive to negative the graph of $y=f(x)$ has relative maximums and when the graph of $y=f'(x)$ has a zero and changes from negative to positive the graph of $y=f(x)$ has a relative minimum.

When the graph of $y=f''(x)$ is positive the graph of $y=f(x)$ is concave up and when the graph of $y=f''(x)$ is negative the graph of $y=f(x)$ is concave down.



Exercise 7V

1 $f(x)=3x^2-18x-48$

$f'(x)=6x-18$

$f'(x)=0 \Rightarrow 6x-18=0 \Rightarrow x=3$

$f''(x)=6 \Rightarrow f''(3)=6>0 \Rightarrow$ relative minimum

$f(3)=-75$

relative minimum : $(3,-75)$

2 $f(x)=(x^2-1)^2$

$f'(x)=2(x^2-1)(2x)=4x(x^2-1)$

$f'(x)=0 \Rightarrow 4x(x^2-1)=0 \Rightarrow x=0,-1,1$

$f''(x)=(4x)(2x)+(x^2-1)(4)=12x^2-4$

$f''(0)=-4<0 \Rightarrow$ relative maximum at $x=0$

$f''(-1)=8>0 \Rightarrow$ relative minimum at $x=-1$

$f''(1)=8>0 \Rightarrow$ relative minimum at $x=1$

$f(0)=1;\ f(-1)=0;\ f(1)=0$

relative maximum: $(0,1)$

relative minimums: $(-1,0)$ and $(1,0)$

3 $f(x)=x^4-4x^3$

$f'(x)=4x^3-12x^2$

$f'(x)=0 \Rightarrow 4x^3-12x^2=0$

$\Rightarrow 4x^2(x-3)=0 \Rightarrow x=0,3$

$f''(x)=12x^2-24x$

$f''(0)=0 \Rightarrow$ second derivative test fails $\Rightarrow$

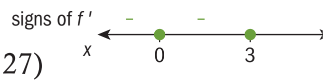
use first derivative test

Since $f'(x)$ does not change signs at $x=0$ there is no relative extremum at $x=0$.

$f''(3)=36>0 \Rightarrow$ relative minimum at $x=3$

$f(3)=-27$

relative minimum: $(3,-27)$



4 $f(x)=xe^x$

$f'(x)=(x)(e^x)+(e^x)(1)=e^x(x+1)$

$f'(x)=0 \Rightarrow e^x(x+1)=0 \Rightarrow x=-1$

$f''(x)=(e^x)(1)+(x+1)(e^x)=e^x(x+2)$

$f''(-1)=\frac{1}{e}>0 \Rightarrow$ relative minimum

$f(-1)=-\frac{1}{e}$

relative minimum: $\left(-1,-\frac{1}{e}\right)$

5 $f(x)=(x-1)^4$

$f'(x)=4(x-1)^3(1)=4(x-1)^3$

$f'(x)=0 \Rightarrow 4(x-1)^3=0 \Rightarrow x=1$

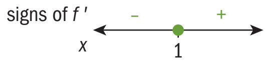
$f''(x)=12(x-1)^2(1)=12(x-1)^2$

$f''(1)=0 \Rightarrow$ second derivative test fails $\Rightarrow$

use first derivative test. $f'(x)$ changes sign from negative to positive at $x=1 \Rightarrow$ relative minimum.

$f(1)=0$

relative minimum: $(1,0)$



6 $f(x)=\frac{1}{x^2+1}=(x^2+1)^{-1}$

$f'(x)=-1(x^2+1)^{-2}(2x)=\frac{-2x}{(x^2+1)^2}$

$f'(x)=0 \Rightarrow \frac{-2x}{(x^2+1)^2}=0 \Rightarrow -2x=0 \Rightarrow x=0$

$$f''(x)=\frac{(x^2+1)^2(-2)-(-2x)\left[2(x^2+1)(2x)\right]}{\left[(x^2+1)^2\right]^2}$$

$$=\frac{-2(x^2+1)\left[(x^2+1)-4x^2\right]}{(x^2+1)^4}=\frac{2(3x^2-1)}{(x^2+1)^3}$$

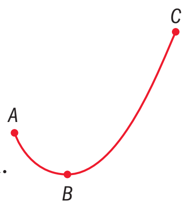
$f''(0)=-2<0 \Rightarrow$ relative maximum

$f(0)=1$

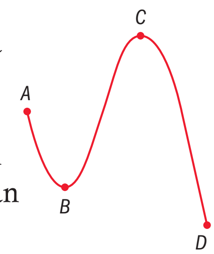
relative maximum : $(0,1)$

Exercise 7W

1 Neither **A** nor **C** can be relative extrema because relative extrema do not occur at endpoints. **A** is neither an absolute nor a relative extrema. **B** is an absolute minimum. **C** is an absolute maximum.



2 Neither **A** nor **D** can be relative extrema because relative extrema do not occur at endpoints. **A** is neither an absolute nor a relative extrema. **B** is a relative minimum **C** is an absolute maximum. **D** is an absolute minimum.



3 $f(x)=(x-2)^3$ for $0 \le x \le 4$

$f'(x)=3(x-2)^2(1)=3(x-2)^2$

$f'(x)=0 \Rightarrow 3(x-2)^2=0 \Rightarrow x=2$

$f(0)=-8$

$f(2)=0$

$f(4)=8$

abolute maximum: 8

abolute minimum: -8

4 $f(x)=8x-x^2$ for $-1\le x\le 7$

$f'(x)=8-2x$

$f'(x)=0\Rightarrow 8-2x=0\Rightarrow x=4$

$f(-1)=-9$

$f(4)=16$

$f(7)=7$

absolute maximum: 16

absolute minimum: -9

5 $f(x)=x^3-\frac{3}{2}x^2$ for $-1\le x\le 2$

$f'(x)=3x^2-3x=3x(x-1)$

$f'(x)=0\Rightarrow 3x(x-1)=0\Rightarrow x=0,1$

$f(-1)=-\frac{5}{2}$

$f(0)=0$

$f(1)=-\frac{1}{2}$

$f(2)=2$

absolute maximum: 2

absolute minimum: $-\frac{5}{2}$

Exercise 7X

1 $x=$ the first positive number

$y=$ the second positive number

$S=x+\sqrt{y}$

$x+y=20\Rightarrow x=20-y$

$S=20-y+\sqrt{y}=20-y+y^{\frac{1}{2}}$

$S'=-1+\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{2\sqrt{y}}-1$

$S'=0\Rightarrow \frac{1}{2\sqrt{y}}-1=0\Rightarrow \frac{1}{2}=\sqrt{y}\Rightarrow y=\frac{1}{4}$

$S''=-\frac{1}{4}y^{-\frac{3}{2}}=\frac{-1}{4\sqrt{y^3}}$

$S''\left(\frac{1}{4}\right)=\frac{-1}{4\sqrt{\left(\frac{1}{4}\right)^3}}=-2<0\Rightarrow$ relative maximum

$x=20-y\Rightarrow x=20-\frac{1}{4}=\frac{79}{4}$

The numbers are $\frac{79}{4}$ and $\frac{1}{4}$

2 $x=$ the first positive number

$y=$ the second positive number

$P=xy$

$x+2y=200\Rightarrow x=200-2y$

$P=(200-2y)(y)=200y-2y^2$

$P'=200-4y$

$P'=0\Rightarrow 200-4y=0\Rightarrow y=50$

$P''=-4$

$P''(50)=-4<0\Rightarrow$ relative maximum

$x=200-2y\Rightarrow x=200-2(50)=100$

The numbers are 100 and 50

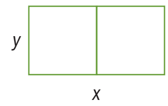
3 $x=$ length

$y=$ width

$A=xy$

$2x+3y=400\Rightarrow x=200-\frac{3}{2}y$

$A=\left(200-\frac{3}{2}y\right)(y)=200y-\frac{3}{2}y^2$



$A'=200-3y$

$A'=0\Rightarrow 200-3y=0\Rightarrow y=\frac{200}{3}$

$A''=-4$

$A''\left(\frac{200}{3}\right)=-4<0\Rightarrow$ relative maximum

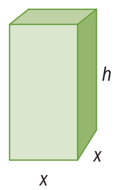
$x=200-\frac{3}{2}y\Rightarrow x=200-\frac{3}{2}\left(\frac{200}{3}\right)=100$

The dimensions are 100 ft by $\frac{200}{3}$ ft.

Exercise 7Y

1 $x=$ length of square base

$h=$ height



$S=x^2+4xh$

$x^2h=32000\Rightarrow h=\frac{32000}{x^2}$

$S=x^2+4x\left(\frac{32000}{x^2}\right)=x^2+128{,}000x^{-1}$

$S'=2x-128{,}000x^{-2}=2x-\frac{128{,}000}{x^2}$

$S'=0\Rightarrow 2x-\frac{128{,}000}{x^2}=0\Rightarrow$

$2x^3=128{,}000\Rightarrow x^3=64000\Rightarrow x=40$

$S''=2+256{,}000x^{-3}=2+\frac{256{,}000}{x^3}$

$S''(40)=6>0\Rightarrow$ relative minimum

$h=\frac{32000}{x^2}\Rightarrow h=\frac{32000}{40^2}=20$

The dimensions are 40 m by 40 m by 20 m.

2 $C(x)=x^3-3x^2-9x+30$

$C'(x)=3x^2-6x-9$

$C'(x)=0\Rightarrow 3x^2-6x-9=0\Rightarrow 3(x+1)(x-3)=0$

$\Rightarrow x=-1,3$

The only ciritcal number in $[0,10]$ is 3.

$C(0)=30$

$C(3)=3$

$C(10)=640$

Three items should be produced to minimize the cost.

3 $s(t)=t^3-12t^2+36t-10$ for $0\le t\le 7$

$s'(t)=3t^2-24t+36$

$s'(t)=0 \Rightarrow 3t^2-24t+36=0 \Rightarrow$

$3(t-2)(t-6)=0 \Rightarrow t=2, 6$

$s(0)=-10$

$s(2)=22$

$s(6)=-10$

$s(7)=-3$

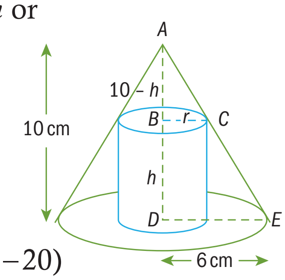
The maximum distance is 22

4 **a** ABC and ADE are similar triangles, so

$\frac{10-h}{r}=\frac{10}{6} \Rightarrow \frac{10-h}{r}=\frac{5}{3} \Rightarrow r=\frac{30-3h}{5}$

b $V=\pi r^2 h \Rightarrow V=\pi\left(\frac{30-3h}{5}\right)^2 h$ or

$\frac{9\pi}{25}(100h-20h^2+h^3)$



c $V=\frac{9\pi}{25}(100h-20h^2+h^3)$

$\frac{dV}{dh}=\frac{9\pi}{25}(100-40h+3h^2)$

$\frac{d^2V}{dh^2}=\frac{9\pi}{25}(-40+6h)=\frac{18\pi}{25}(3h-20)$

d $\frac{dV}{dh}=0 \Rightarrow \frac{9\pi}{25}(100-40h+3h^2)=0 \Rightarrow$

$\frac{9\pi}{25}(10-3h)(10-h) \Rightarrow h=\frac{10}{3}, 10$

$\left.\frac{d^2V}{dh^2}\right|_{h=\frac{10}{3}}=\frac{-180\pi}{25}<0 \Rightarrow$ relative maximum

$\left.\frac{d^2V}{dh^2}\right|_{h=10}=\frac{180\pi}{25}>0 \Rightarrow$ relative minimum

$r=\frac{30-3h}{5} \Rightarrow r=\frac{30-3\left(\frac{10}{3}\right)}{5}=4$

Maximum volume occurs when the radius is 4in and the height is $\frac{10}{3}$in.

5 **a** $p(x)=4\sqrt{x}-2x^2$

b $p(x)=4\sqrt{x}-2x^2=4x^{\frac{1}{2}}-2x^2$

$\frac{dp}{dx}=2x^{-\frac{1}{2}}-4x=\frac{2}{\sqrt{x}}-4x$

$\frac{d^2p}{dx^2}=-x^{-\frac{3}{2}}-4=\frac{-1}{\sqrt{x^3}}-4$

c $\frac{dp}{dx}=0 \Rightarrow \frac{2}{\sqrt{x}}-4x=0 \Rightarrow x\approx 0.630$

$\left.\frac{d^2p}{dx^2}\right|_{x\approx 0.630}=-6<0 \Rightarrow$ relative maximum

0.630 thousand units or 630 units maximize the profit.

Review exercise

1 **a** $\frac{d}{dx}\left(4x^3+3x^2-2x+6\right)=12x^2+6x-2$

b $\frac{d}{dx}\left(\sqrt[3]{x^4}\right)=\frac{d}{dx}\left(x^{\frac{4}{3}}\right)=\frac{4}{3}x^{\frac{1}{3}}$

c $\frac{d}{dx}\left(\frac{3}{x^4}\right)=\frac{d}{dx}\left(3x^{-4}\right)=-12x^{-5}=\frac{-12}{x^5}$

d $\frac{d}{dx}\left[(x^2-1)(2x^3-x^2+x)\right]$

$=(x^2-1)(6x^2-2x+1)+(2x^3-x^2+x)(2x)$

$=\left(6x^4-2x^3+x^2-6x^2+2x-1\right)$

$+\left(4x^4-2x^3+2x^2\right)$

$=10x^4-4x^3+3x^2+2x-1$

e $\frac{d}{dx}\left[\frac{x-4}{x+7}\right]=\frac{(x+7)(1)-(x-4)(1)}{(x+7)^2}=\frac{11}{(x+7)^2}$

f $\frac{d}{dx}\left[e^{4x}\right]=(e^{4x})(4)=4e^{4x}$

g $\frac{d}{dx}\left[(x^3+1)^4\right]=4(x^3+1)^3(3x^2)=12x^2(x^3+1)^3$

h $\frac{d}{x}\left[\ln(2x+3)\right]=\left(\frac{1}{2x+3}\right)(2)=\frac{2}{2x+3}$

i $\frac{d}{dx}\left[\frac{\ln x}{x^2}\right]=\frac{(x^2)\left(\frac{1}{x}\right)-(\ln x)(2x)}{(x^2)^2}=\frac{x-2x\ln x}{x^4}=\frac{1-2\ln x}{x^3}$

j $\frac{d}{dx}\left[\frac{4x^2-2x}{6}\right]=\frac{d}{dx}\left[\frac{1}{6}(4x^2-2x)\right]=\frac{1}{6}(8x-2)=\frac{4}{3}x-\frac{1}{3}$

k $\frac{d}{dx}\left[(3x^2+1)(e^x)\right]=(3x^2+1)(e^x)+(e^x)(6x)$

$=e^x(3x^2+6x+1)$

l $\frac{d}{dx}\left[\frac{2e^x}{e^x-3}\right]=\frac{(e^x-3)(2e^x)-(2e^x)(e^x)}{(e^x-3)^2}=\frac{-6e^x}{(e^x-3)^2}$

m $\frac{d}{dx}\left[3\sqrt{2x-5}\right]=\frac{d}{dx}\left[3(2x-5)^{\frac{1}{2}}\right]=\frac{3}{2}(2x-5)^{-\frac{1}{2}}(2)$

$=\frac{3}{\sqrt{2x-5}}$

n $\frac{d}{dx}\left[x^2e^{2x}\right]=(x^2)\left[(e^{2x})(2)\right]+(e^{2x})(2x)$

$=2xe^{2x}(x+1)$

o $\frac{d}{dx}\left[\ln\left(\frac{1}{x}\right)\right]=\frac{d}{dx}\left[\ln x^{-1}\right]=\left(\frac{1}{x^{-1}}\right)(-x^{-2})$

$=(x)\left(\frac{-1}{x^2}\right)=\frac{-1}{x}$

2 **a** $(x+h)^3=x^3+3x^2h+3xh^2+h^3$

b $f'(x)=\lim_{h\to 0}\frac{[2(x+h)^3-6(x+h)]-(2x^3-6x)}{h}$

$=\lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3-6x-6h-2x^3+6x}{h}$

$=\lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3-6h}{h}$

$=\lim_{h\to 0}\frac{h(6x^2+6xh+2h^2-6)}{h}$

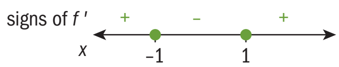
$=\lim_{h\to 0}(6x^2+6xh+2h^2-6)$

$=6x^2-6$

c $f'(x)=6x^2-6$

$f'(x)=0 \Rightarrow 6x^2-6=0 \Rightarrow 6(x+1)(x-1)=0$

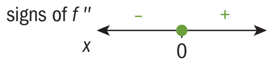
$\Rightarrow x=-1, 1$



$f'(x)<0$ for $-1<x<1$, so $p=-1$ and $q=1$.

d $f''(x) = 12x$

e $f''(x) = 0 \Rightarrow 12x = 0 \Rightarrow x = 0$



$f''(x) > 0$ for $x > 0$ so f is concave up on $(0, \infty)$.

3 $f(x) = 4xe^{x^2-1}$; normal line at $(1, 4)$

$f'(x) = (4x)\left[(e^{x^2-1})(2x)\right] + (e^{x^2-1})(4)$

$= 2e^{x^2-1}(4x^2 + 2)$

$m_{\text{tangent}} = f'(1) = 12$

$m_{\text{normal}} = -\frac{1}{12}$

$y - 4 = -\frac{1}{12}(x - 1)$

4 $f(x) = 2x^3 - 3x + 1$; tangent line parallel to $y = 5x - 2$

$f'(x) = 6x^2 - 3$

$6x^2 - 3 = 5 \Rightarrow x = \pm\frac{2\sqrt{3}}{3}$

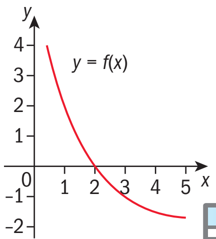
$f\left(\frac{2\sqrt{3}}{3}\right) = \frac{9 - 2\sqrt{3}}{9}$, $f\left(\frac{-2\sqrt{3}}{3}\right) = \frac{9 + 2\sqrt{3}}{9}$

The tangent line is parallel to $y = 5x + 2$ at the points

$\left(\frac{2\sqrt{3}}{3}, \frac{9 - 2\sqrt{3}}{9}\right)$ and $\left(\frac{-2\sqrt{3}}{3}, \frac{9 + 2\sqrt{3}}{9}\right)$

5 **a** $f''(2) > f(2) > f'(2)$

b $f''(2) > 0$ since the graph of f is concave up, $f(2) = 0$ and $f'(2) < 0$ since the graph of f is decreasing



6 **a** **i** $y = x^3(x - 4) = x^4 - 4x^3$

$\frac{dy}{dx} = 4x^3 - 12x^2$

ii $\frac{d^2y}{dx^2} = 12x^2 - 24x$

b **i** $x^3(x - 4) = 0 \Rightarrow x = 0, 4 \Rightarrow$ x-intercepts are $(0, 0)$ and $(4, 0)$

ii $\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x - 3)$

$\frac{dy}{dx} = 0 \Rightarrow 4x^2(x - 3) = 0 \Rightarrow x = 0, 3$

$\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x - 2)$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$

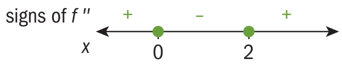
$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 36 > 0 \Rightarrow$ relative minimum

$y(3) = 3^3(3 - 4) = -27$

relative minimum point: $(3, -27)$

iii $\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x - 2)$

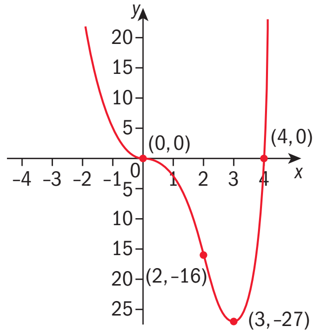
$\frac{d^2y}{dx^2} = 0 \Rightarrow 12x(x - 2) = 0 \Rightarrow x = 0, 2$



$y(0) = 0$; $y(2) = 2^3(2 - 4) = -16$

inflexion points: $(0, 0)$ and $(2, -16)$

c

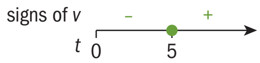


7 **a** $s(t) = 20t - 100\ln t$, $t \geq 1$.

$v(t) = s'(t)$

$v(t) = 20 - \frac{100}{t}$

b $v(t) = 0 \Rightarrow 20 - \frac{100}{t} = 0 \Rightarrow t = 5$



The particle is moving to the left on the interval $(1, 5)$

c $v(t) = 20 - \frac{100}{t} = 20 - 100t^{-1}$

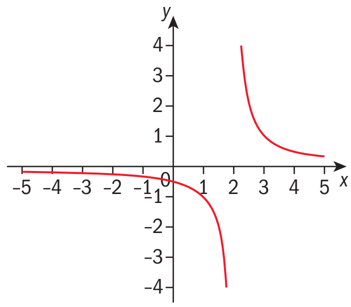
$v'(t) = a(t) = 100t^{-2} = \frac{100}{t^2}$

Since $100 > 0$ and $t^2 > 0$, $v'(t) > 0$ for all $t \geq 1$.

Therefore velocity is always increasing.

Review exercise

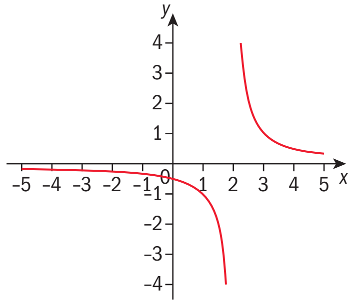
1 **a** $\lim_{x \to 2} \frac{1}{x - 2}$



x	$f(x)$
1.5	–2.0000
1.6	–2.5000
1.7	–3.3333
1.8	–5.0000
1.9	–10.0000
2.0	
2.1	10.0000
2.2	5.0000
2.3	3.3333
2.4	2.5000
2.5	2.0000

$\lim_{x \to 2} \frac{1}{x - 2}$ does not exist at 2 since the function approaches $-\infty$ from the left side of 2 and ∞ on the right side of 2.

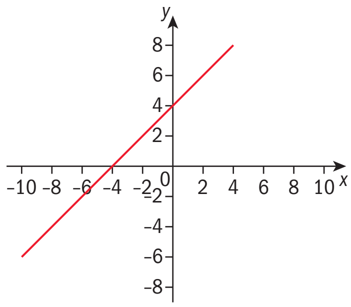
b $\lim_{x\to 3}\frac{1}{x-2}$



x	f(x)
2.5	2.0000
2.6	1.6667
2.7	1.4286
2.8	1.2500
2.9	1.1111
3.0	1.0000
3.1	0.9091
3.2	0.8333
3.3	0.7692
3.4	0.7143
3.5	0.6667

$\lim_{x\to 3}\frac{1}{x-2}=\frac{1}{3-2}=1$

c $\lim_{x\to 4}\frac{x^2-16}{x-4}$

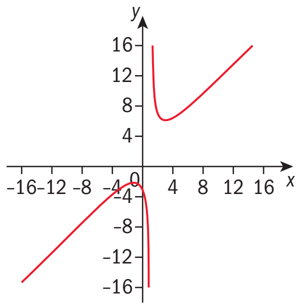


x	f(x)
3.5	7.5000
3.6	7.6000
3.7	7.7000
3.8	7.8000
3.9	7.9000
4.0	
4.1	8.1000
4.2	8.2000
4.3	8.3000
4.4	8.4000
4.5	8.5000

$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}$

$=\lim_{x\to 4}(x+4)=4+4=8$

d $\lim_{x\to 1}\frac{x^2+3}{x-1}$



x	f(x)
0.5	−6.5000
0.6	−8.4000
0.7	−11.6333
0.8	−18.2000
0.9	−38.1000
1.0	
1.1	42.1000
1.2	22.2000
1.3	15.6333
1.4	12.4000
1.5	10.5000

$\lim_{x\to 1}\frac{x^2+3}{x-1}$ does not exist at 1 since the function approaches $-\infty$ from the left side of 1 and ∞ on the right side of 1.

2 a i $x^2+10^2=y^2 \Rightarrow y=\sqrt{x^2+100}$

ii $(30-x)^2+25^2=z^2 \Rightarrow$

$z=\sqrt{(30-x)^2+625} \Rightarrow$

$z=\sqrt{x^2-60x+1525}$

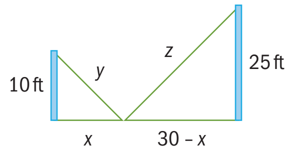
iii $L(x)=y+z=\sqrt{x^2+100}+\sqrt{x^2-60x+1525}$

$=(x^2+100)^{\frac{1}{2}}+(x^2-60x+1525)^{\frac{1}{2}}$

b i $L'(x)=\frac{1}{2}(x^2+100)^{-\frac{1}{2}}(2x)+$

$\frac{1}{2}(x^2-60x+1525)^{-\frac{1}{2}}(2x-60)$

$=\frac{x}{\sqrt{x^2+100}}+\frac{x-30}{\sqrt{x^2-60x+1525}}$



ii $L'(x)=0 \Rightarrow \frac{x}{\sqrt{x^2+100}}+\frac{x-30}{\sqrt{x^2-60x+1525}}$

$=0 \Rightarrow x\approx 8.57$

Signs of $L'(x)$

$L(x)$ has a minimum at $x = 8.57$

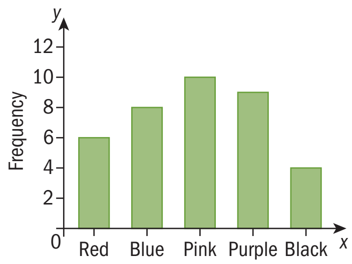
The stake should be placed 8.47 ft from the 10 foot post.

8 Descriptive statistics

Answers

Skills check

1



2 **a** Mean $= \frac{4+7+7+8+6}{5} = \frac{32}{5} = 6.4$

b The number that occurs most often is 8

c **i** Arrange the data in order of size. 2, 4, 4, 6, 7, 8, 11. The median is the middle member, 6

ii 5, 7, 9, 11, 13, 15. The middle member is in between 9 and 11. $\frac{1}{2}(9 + 11) = 10$

iii 6, 8, 11, 11, 14, 17. The middle member is between the two number elevens. 11

Exercise 8A

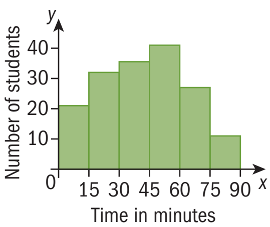
1 **a** Discrete. **b** Continuous.
c Continuous. **d** Discrete

2 Discrete

Exercise 8B

1 **a** Continuous

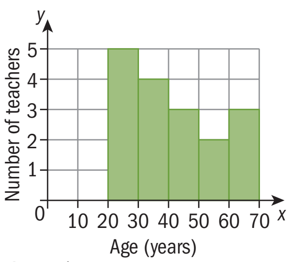
b



2 **a** Continuous

b 17

c



3 **a** Continuous

b

Weight (kg)	$1 \le w < 2$	$2 \le w < 3$	$3 \le w < 4$	$4 \le w < 5$
Number of chickens	8	24	50	14

c 8 + 24 + 50 + 14 = 96

4 **a** Continuous

b

Time	*f*
$5 \le t < 10$	1
$10 \le t < 15$	2
$15 \le t < 20$	4
$20 \le t < 25$	4
$25 \le t < 30$	2
$30 \le t < 35$	2
$35 \le t < 40$	1
$40 \le t < 45$	1

c 5 mins

Exercise 8C

1 **a** 18 **b** 9 **c** 18 and 24
d 0 **e** $\frac{1}{2}$ and 2.

2 **a** 1 **b** $170 \le t < 180$

Exercise 8D

1 Mean $= \frac{\sum x}{n} = \frac{66+57+71+69+58+54}{6} = \frac{375}{6} = 62.5\,\text{kmh}^{-1}$

2 Mean $= \frac{\sum x}{n} = \frac{1.79+1.61+1.96+2.08}{4} = \frac{7.44}{4} = \$1.86$

3 **a** Discrete

b

Calls per day (*x*)	*f*	*fx*
2	3	6
3	2	6
4	5	20
5	3	15
6	4	24
7	6	42
8	3	24
9	4	36
Totals	30	173

Mean $= \frac{\sum fx}{\sum f} = \frac{173}{30} = 5.7\dot{6}$ calls per day.

4 **a** Continuous **b** $90 \le m < 120$

c

Minutes (*m*)	*f*	Midpoint(*m*)	*fm*
$0 \le m < 30$	12	15	180
$30 \le m < 60$	16	45	720
$60 \le m < 90$	20	75	1500
$90 \le m < 120$	36	105	3780
$120 \le m < 150$	16	135	2160
Totals	100		8340

Mean $= \frac{\sum fm}{\sum f} = \frac{8340}{100} = 83.4$ minutes per day.

5 Let x be Kelly's score on the fifth test. To average 84

$84 = \frac{95+82+76+88+x}{5}$

$84 \times 5 = 341 + x$

$420 = 341 + x$

$x = 420 - 341$

$x = 79$

Kelly must score 79 on the next test.

6 Total mass of 11 players = 11 × 80.3 = 883.3 kg

$81.2 = \frac{883.3+x}{12}$

$81.2 \times 12 = 883.3 + x$

$974.4 = 883.3 + x$

$x = 974.4 - 883.3$

$x = 91.1\text{kg}$

The new player has mass 91.1 kg

7 Let x be the distance they travel on sixth day.

$250 = \frac{220+300+210+275+240+x}{6}$

$250 \times 6 = 1245 + x$

$1500 = 1245 + x$

$x = 1500 - 1245$

$x = 255$

They must travel 255 km on the last day.

8 Total number of shots = 8 × 71 = 568.

9 After 8 matches, the total points scored is 8 × 27 = 216

After 11 matches the total points scored is 11 × 29 = 319

319–216 = 103 points scored in the last 3 matches

10 Billy's total = 12 × 310 = $3720

Jean's total = 13 × 320 = $4160

Billy's total + Jean's total = $7880

Mean = $\frac{7880}{25}$ = $315.20

Exercise 8E

1 **a** Arrange in order of size 2, 2, 3, 3, 4, 4, 5, 6, 7

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{9+1}{2}\right)^{th} = 5^{th} = 4$

b 2, 2, 3, 5, 5, 7, 8

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{7+1}{2}\right)^{th} = 4^{th} = 5$

c 0, 2, 3, 3, 4, 6, 7, 9

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{8+1}{2}\right)^{th} = 4.5^{th} = 3.5$

d 0, 1, 1.5, 2, 4, 4, 5, 8, 8.4, 9, 12

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{11+1}{2}\right)^{th} = 6^{th} = 4$

e 1, 2, 4, 5, 7, 9, 12, 20

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{8+1}{2}\right)^{th} = 4.5^{th} = 6$

2 Number of CDs= 3+2+2+1+3+5+3 = 19

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{19+1}{2}\right)^{th} = 10^{th} = 11$

3 Mean = 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 9, 10.

Mean $= \frac{\text{sum of scores}}{\text{Number of students}} = \frac{168}{32} = 5.25$

Mode = 7

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{32+1}{2}\right)^{th} = 16.5^{th} = 5.5$

Investigation – measures of central tendency

	Data	Mean	Mode	Median
Data Set	6, 7, 8, 10, 12, 14, 14, 15, 16, 20	12.2	14	13
Add four to each data set	10, 11, 12, 14, 16, 18, 18, 19, 20, 24	16.2	18	17
Multiply the original data set by 2	12, 14, 16, 20, 24, 28, 28, 30, 32, 40	24.4	28	26

a If you add 4 to each data value, you will add 4 to the mean, mode and median.

b If you multiply each data value by 2, you will multiply the mean, mode and median by 2.

Exercise 8F

1 **a** Range = largest – smallest = 125–30 = 95 cm

b In ascending order, depths are 30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125

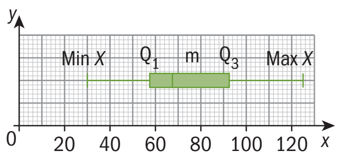
Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{12+1}{2}\right)^{th} = 6.5^{th} = 67.5$

c $Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(12+1)^{th} = 3.25^{th} = 57.5$

d $Q_3 = \frac{3}{4}(n+1)^{th} = \frac{3}{4}(12+1)^{th} = 9.75^{th} = 92.5$

e $IQR = Q_3 - Q_1 = 92.5 - 57.5 = 35$

f



2 **a** Range = largest–smallest = 85 – 71 = 14

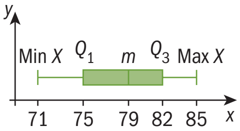
b Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{11+1}{2}\right)^{th} = 6^{th} = 79$

c $Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(11+1)^{th} = 3^{rd} = 75$

d $Q_3 = \frac{3}{4}(n+1)^{th} = \frac{3}{4}(11+1)^{th} = 9^{th} = 82$

e $IQR = Q_3 - Q_1 = 82 - 75 = 7$

f



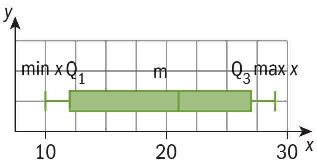
3 **a** Range = largest-smallest = 29–10 = 19

b Median $=\left(\frac{n+1}{2}\right)^{th}=\left(\frac{11+1}{2}\right)^{th}=6^{th}=21$

c $Q_1=\frac{1}{4}(n+1)^{th}=\frac{1}{4}(11+1)^{th}=3^{rd}=12$

d $Q_3=\frac{3}{4}(n+1)^{th}=\frac{3}{4}(11+1)^{th}=9^{th}=27$

e $IQR=Q_3-Q_1=27-12=15$



4 **a** Range = 11–6 = 5 **b** 8 **c** 7

d 10 **e** IQR = 10–7 = 3

5 Histogram **i**.

Shows range of 10–1 = 9.

Shows sample size of 6 + 6 + 6 + 6 + 6 = 30

Median $=\left(\frac{30+1}{2}\right)^{th}=15.5^{th}$ which lies in 5–6 category

$Q_1=\frac{1}{4}(30+1)^{th}=7.75^{th}$ which lies in 3–4 category

$Q_2=\frac{3}{4}(30+1)^{th}=23.25^{th}$ which lies in 7–8 category

This information is shown by box plot **c**.

Histogram **ii**.

Shows range 10–1 = 9.

Shows sample size of 4.5 + 7 + 8 + 4.5 + 3.5 = 27.5.

Median $=\left(\frac{27.5+1}{2}\right)^{th}=14.25^{th}$ which lies in 5–6 category.

$Q_1=\frac{1}{4}(27.5+1)^{th}=7.125^{th}$ lies in 3–4 category

$Q_2=\frac{3}{4}(27.5+1)^{th}=20.625^{th}$ lies in 7–8 category

This is shown in box plot **b**.

Histogram **iii**.

Shows range 10–1 = 9.

Sample size: 7.5 + 2.5 + 5.5 + 3.5 + 8 = 27

Median $=\left(\frac{27+1}{2}\right)^{th}=14^{th}$ lies in 5–6 category

$Q_1=\frac{1}{4}(27+1)^{th}=7^{th}$ lies in 1–2 category

$Q_3=\frac{3}{4}(27+1)^{th}=21^{th}$ lies in 9–10 category

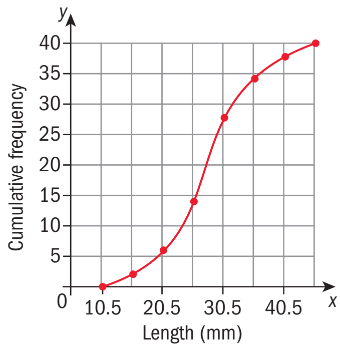
This is shown in box plot **a**.

Exercise 8G

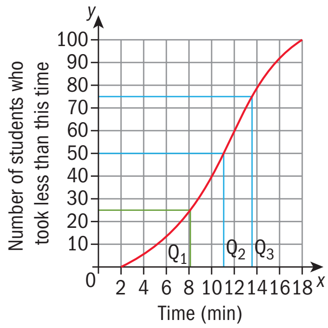
1 **a** 75 cm. **b** (77.5–72) cm = 5.5 cm.

c Half of the boxers have a reach of 72 cm to 77.5 cm.

2



3



i 11 mins

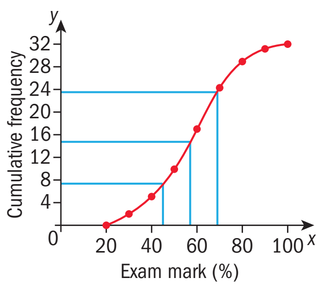
ii (13.6–8.2) mins = 5.4 mins.

b p = 32, q = 8.

4 **a**

Marks	*f*	*CF*
20 ≤ *m* < 30	2	2
30 ≤ *m* < 40	3	5
40 ≤ *m* < 50	5	10
50 ≤ *m* < 60	7	17
60 ≤ *m* < 70	6	23
70 ≤ *m* < 80	4	27
80 ≤ *m* < 90	2	29
90 ≤ *m* < 100	1	30

b



c **i** Median ≈ 57%

ii Lower quartile ≈ 45%

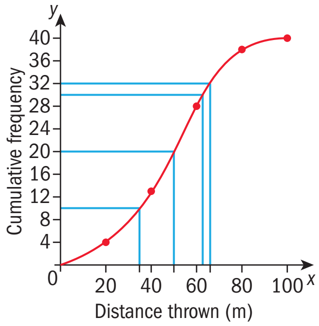
Upper quartile ≈ 69%

iii Interquartile range ≈ 69% - 45% = 24%

5 **a**

Distance (*m*)	*f*	*CF*
0 ≤ *m* < 20	4	4
20 ≤ *m* < 40	9	13
40 ≤ *m* < 60	15	28
60 ≤ *m* < 80	10	38
80 ≤ *m* < 100	2	40

b



c 20% of 40 students = 8.

Only top 8 students will made final.

40 - 8 = 32.

We draw a line across from 32 on y-axis, and down to see the required distance.

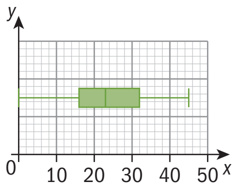
Qualifying distance ≈ 66 m.

d Interquartile range ≈ 63 - 35 = 28

e Median ≈ 50 m

6 a i 23 mins **ii** 16 mins **iii** 36 mins

b



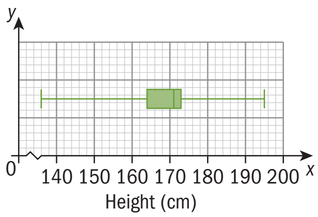
7 a 171 cm

b 55 flowers between 135 cm and 164 cm

c 22 flowers. 181 cm

d 110.

e



Exercise 8H

1 a mean = 18, variance = 129.6, standard deviation = 11.4

b mean = 40, variance = 200, standard deviation = 14.1

2 a variance = 78.5 standard deviation = 8.86

b variance = 80.19 standard deviation = 8.95

c variance = 449 standard deviation = 21.2

3 1.32

4 mean = 2.5, standard deviation = 1.22

5 mean = 26.2, standard deviation = 14.9

6 a discrete **b** 2.73 **c** 1.34 **d** 23

7 mean = 42.4, standard deviation = 21.6

8 a 51 **b** 69.5 **c i** 21.8 **ii** none

Investigation – the effect of adding or multiplying the data set of a standard deviation

a 2.47

b The mean has had 100 added to it.

c 2.47

d The standard deviation remains the same. This is because the standard deviation only measures the spread of the numbers, and that remains constant if the same number is added to each item in the list.

e The mean is doubled.

f 4.94

g The variance will be multiplied by 4 because the variance is the standard deviation squared.

Review exercise

1 a Mode = 3 as 3 appears the most in the list.

b First write the numbers in ascending order: 1, 2, 3, 3, 5, 6, 7, 8, 10

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} = \left(\frac{9+1}{2}\right)^{th} = 5^{th} = 5.$$

c $\text{Mean} = \dfrac{1+2+3+3+5+6+7+8+10}{9} = 5$

d Range = 10 - 1 = 9.

2 a

Pets (p)	2	3	4	5	6	7	8	9	10
f	3	9	10	2	3	1	1	0	1
fp	6	27	40	10	18	7	8	0	10

$$\text{Mean} = \frac{\Sigma\,\text{fp}}{\Sigma\,\text{f}} = \frac{126}{30} = 4.2.$$

b $\text{Median} = \left(\frac{n+1}{2}\right)^{th}$ value $= \left(\frac{30+1}{2}\right)^{th} = 15.5^{th} = 4$

c Mode = 4.

3 Mean = 27.5 yrs, standard deviation = 0.4 yrs.

4 Type A:

a Median = 52

b Range 60 - 46 = 14

c Inter Quartile range = 57 - 49 = 8.

Type B:

a Median = 52

b Range = 57 - 49 = 8

c Inter Quartile range = 54 - 51 = 3.

5 **a** $\frac{46 + 92 + 4x}{6} = 71.$

$4x = (6 \times 71) - 138$

$x = \frac{288}{4} = 72.$

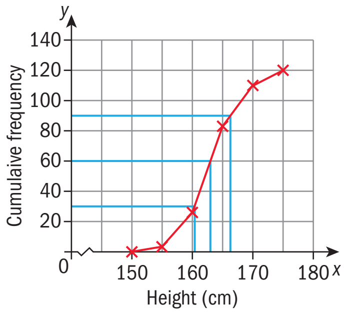
So total = 46 + 92 + 4(72) = 426

b $x = 72$. (from part **a**).

c New mean decreased by 9 also.
New mean is 71 - 9 = 62

6 **a**

Height	*f*	Σ*f*
$150 \le h < 155$	4	4
$155 \le h < 160$	22	26
$160 \le h < 165$	56	82
$165 \le h < 170$	32	114
$170 \le h < 175$	5	119



b Median ≈ 163,

c IQR ≈ 6.

7 **a** $26 + 10 + 20 + k + 29 + 11 = 100$

$k = 100 - 96$

$k = 4$

b **i** Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{100+1}{2}\right) = 50.5^{th} = 3$

ii $Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(100+1)^{th} = 25.25^{th} = 1$

$Q_3 = \frac{3}{4}(n+1)^{th} = \frac{3}{4}(100+1)^{th} = 75.75^{th} = 5$

Interquartile range = 5 - 1 = 4.

8 Total readings = 6 + 3 + 5 + 8 + 6 + 2 = 30

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{30+1}{2}\right) = 15.5^{th} = 65$

$Q_1 = \frac{1}{4}(n+1)^{th} = \frac{1}{4}(31) = 7.75 = 35$

$Q_2 = \frac{3}{4}(n+1)^{th} = \frac{3}{4}(31) = 23.25 = 80$

IQR = 80 - 35 = 45.

Note: 65 is the midpoint of $57.5 \le t \le 72.5$, in which the 15.5^{th} value lies.

Review exercise

1 Median = 20 IQR = 14.

2 **a** 6.48

b 1.31

3 **a** 6

b 6

c 5.92

4 **a** Mean = 2.57, median = 2, mode = 1, standard deviation = 1.68 and variance = 2.82.

b Range = 6, lower quartile = 1 and the interquartile range = 3.

5 **a** $160 \le Height < 170$

b

Height	*f*
$140 \le Height < 150$	15
$150 \le Height < 160$	55
$160 \le Height < 170$	90
$170 \le Height < 180$	45
$180 \le Height < 190$	5

Mean = 163.6 cm

6 **a** **i** $p = 65$

ii $q = 34$

b median = 18

c mean = 17.7

9 Integration

Answers

Skills check

1 a $\displaystyle\sum_{i=1}^{5}(2i^2) = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2$

$= 2 + 8 + 18 + 32 + 50$

$= 110$

b $\displaystyle\sum_{k=2}^{6}(3k-2) = [3(2)-2] + [3(3)-2] + [3(4)-2]$

$+ [3(5)-2] + [3(6)-2]$

$= 4 + 7 + 10 + 13 + 16$

$= 50$

c $\displaystyle\sum_{i=1}^{5}\left[(i)^2 g(x_i)\right] = \left[(1)^2 g(x_1)\right] + \left[(2)^2 g(x_2)\right]$

$+ \left[(3)^3 g(x_3)\right] + \left[(4)^4 g(x_4)\right]$

$+ \left[(5)^5 g(x_5)\right]$

$= g(x_1) + 4g(x_2) + 9g(x_3)$

$+ 16g(x_4) + 25g(x_5)$

d $\displaystyle\sum_{j=1}^{3}\left[f(x_j)(\Delta x_j)\right] = f(x_1)(\Delta x_1) + f(x_2)(\Delta x_2)$

$+ f(x_3)(\Delta x_3)$

2 a $A = \frac{1}{2}b \times h = \frac{1}{2} \times 9 \times 4 = 18 \text{ mm}^2$

b $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4)^2 = 8\pi \text{ cm}^2$

3 a $V = \pi r^2 h = \pi(4)^2(10) = 160\pi \text{ m}^3$

b $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3)^2(14) = 42\pi\, ft^3$

Exercise 9A

1 $\frac{1}{7+1}x^{7+1} + C = \frac{1}{8}x^8 + C$

2 $\frac{1}{4+1}x^{4+1} + C = \frac{1}{5}x^5 + C$

3 $\frac{1}{-2+1}x^{-2+1} + C = -1x^{-1} + C = -\frac{1}{x} + C$

4 $\left(\frac{1}{-\frac{1}{2}+1}\right)x^{-\frac{1}{2}+1} + C = \left(\frac{1}{\frac{1}{2}}\right)x^{\frac{1}{2}} + C = 2x^{\frac{1}{2}} + C$

5 $\left(\frac{1}{\frac{1}{3}+1}\right)x^{\frac{1}{3}+1} + C = \left(\frac{1}{\frac{4}{3}}\right)x^{\frac{4}{3}} + C = \frac{3}{4}x^{\frac{4}{3}} + C$

6 $\left(\frac{1}{\frac{2}{5}+1}\right)x^{\frac{2}{5}+1} + C = \left(\frac{1}{\frac{7}{5}}\right)x^{\frac{7}{5}} + C = \frac{5}{7}x^{\frac{7}{5}} + C$

7 $\frac{1}{x^4} = x^{-4};\ \frac{1}{-4+1}x^{-4+1} + C = -\frac{1}{3}x^{-3} + C = -\frac{1}{3x^3} + C$

8 $\frac{1}{x^{12}} = x^{-12};\ \frac{1}{-12+1}x^{-12+1} + C = -\frac{1}{11}x^{-11} + C = -\frac{1}{11x^{11}} + C$

9 $\sqrt[3]{x} = x^{\frac{1}{3}};\ \left(\frac{1}{\frac{1}{3}+1}\right)x^{\frac{1}{3}+1} + C = \left(\frac{1}{\frac{4}{3}}\right)x^{\frac{4}{3}} + C = \frac{3}{4}x^{\frac{4}{3}} + C$

10 $\sqrt[7]{x^3} = x^{\frac{3}{7}};\ \left(\frac{1}{\frac{3}{7}+1}\right)x^{\frac{3}{7}+1} + C = \left(\frac{1}{\frac{10}{7}}\right)x^{\frac{10}{7}} + C = \frac{7}{10}x^{\frac{10}{7}} + C$

11 $\frac{1}{\sqrt[5]{x}} = x^{-\frac{1}{5}};\ \left(\frac{1}{-\frac{1}{5}+1}\right)x^{-\frac{1}{5}+1} + C = \left(\frac{1}{\frac{4}{5}}\right)x^{\frac{4}{5}} + C = \frac{5}{4}x^{\frac{4}{5}} + C$

12 $\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}};\ \left(\frac{1}{-\frac{2}{3}+1}\right)x^{-\frac{2}{3}+1} + C = \left(\frac{1}{\frac{1}{3}}\right)x^{\frac{1}{3}} + C = 3x^{\frac{1}{3}} + C$

Exercise 9B

1 $\int x^3 dx = \frac{1}{3+1}x^{3+1} + C = \frac{1}{4}x^4 + C$

2 $\int \frac{1}{t^2}dt = \int t^{-2}dt = \frac{1}{-2+1}t^{-2+1} + C = -t^{-1} + C = -\frac{1}{t} + C$

3 $\int \sqrt[5]{x^4}\,dx = \int x^{\frac{4}{5}}dx = \frac{1}{\frac{4}{5}+1}x^{\frac{4}{5}+1} + C = \frac{5}{9}x^{\frac{9}{5}} + C$

4 $\int 2du = 2u + C$

5 $\int (3x^2 + 2x + 1)dx = 3\int x^2 dx + 2\int x dx + \int 1 dx$

$= 3\left(\frac{1}{2+1}x^{2+1}\right) + 2\left(\frac{1}{1+1}x^{1+1}\right) + x + C$

$= x^3 + x^2 + x + C$

6 $\int \frac{4}{x^3}dx = \int 4x^{-3}dx = 4\int x^{-3}dx = 4\left(\frac{1}{-3+1}x^{-3+1}\right) + C$

$= -2x^{-2} + C = -\frac{2}{x^2} + C$

7 $\int \left(t^2 + \sqrt[4]{t}\right)dt = \int t^2 dt + \int t^{\frac{1}{4}}dt$

$= \frac{1}{2+1}t^{2+1} + \frac{1}{\frac{1}{4}+1}t^{\frac{1}{4}+1} + C = \frac{1}{3}t^3 + \frac{4}{5}t^{\frac{5}{4}} + C$

8 $\int \left(\sqrt[3]{x^2} + 1\right)dx = \int x^{\frac{2}{3}}dx + \int 1 dx = \frac{1}{\frac{2}{3}+1}x^{\frac{2}{3}+1} + x + C$

$= \frac{3}{5}x^{\frac{5}{3}} + x + C$

9 $\int(5x^4+12x^3+6x-2)dx=5\int x^4dx+12\int x^3dx$

$+6\int xdx-\int 2dx$

$=5\left(\frac{1}{4+1}x^{4+1}\right)+12\left(\frac{1}{3+1}x^{3+1}\right)$

$+6\left(\frac{1}{1+1}x^{1+1}\right)-2x+C$

$=x^5+3x^4+3x^2-2x+C$

10 $\int dt=\int 1dt=t+C$

11 a $f(x)=x^3+\frac{4}{x^2}=x^3+4x^{-2};\ f'(x)=3x^{3-1}+4\left(-2x^{-2-1}\right)$

$=3x^2-8x^{-3}=3x^2-\frac{8}{x^3}$

b $\int f(x)dx=\int\left(x^3+4x^{-2}\right)dx=\int x^3dx+4\int x^{-2}dx$

$=\frac{1}{3+1}x^{3+1}+4\left(\frac{1}{-2+1}x^{-2+1}\right)+C$

$=\frac{1}{4}x^4-\frac{4}{x}+C$

12 a $g(x)=30\sqrt[5]{x}=30x^{\frac{1}{5}};\ g'(x)=30\left(\frac{1}{5}x^{\frac{1}{5}-1}\right)=\frac{6}{x^{\frac{4}{5}}}$

b $\int g(x)dx=30\int x^{\frac{1}{5}}dx=30\left(\frac{1}{\frac{1}{5}+1}x^{\frac{1}{5}+1}\right)+C$

$=30\left(\frac{5}{6}x^{\frac{6}{5}}\right)+C=25x^{\frac{6}{5}}+C$

Exercise 9C

1 $f(x)=\int(4x^5+8x)dx=\frac{2}{3}x^6+4x^2+C$

$f(0)=8=\frac{2}{3}(0)^6+4(0)^2+C$

$C=8$

$f(x)=\frac{2}{3}x^6+4x^2+8$

2 $y=\int\left(x^4+\sqrt[4]{x}\right)dx=\int\left(x^4+x^{\frac{1}{4}}\right)dx=\frac{1}{5}x^5+\frac{4}{5}x^{\frac{5}{4}}+C$

$10=\frac{1}{5}(1)^5+\frac{4}{5}(1)^{\frac{5}{4}}+C$

$C=9$

$y=\frac{1}{5}x^5+\frac{4}{5}x^{\frac{5}{4}}+9$

3 $s(t)=\int v(t)dt$

$s(t)=\int(3t^2-2t)dt=t^3-t^2+C$

$12=3^3-3^2+C$

$C=-6$

$s(t)=t^3-t^2-6$

4 $V(t)=\int 2\pi(4t^2+4t+1)dt=2\pi\int(4t^2+4t+1)dt$

$=2\pi\left(\frac{4}{3}t^3+2t^2+t\right)+C$

$\pi=2\pi\left(\frac{4}{3}(0)^3+2(0)^2+0\right)+C$

$C=\pi$

$V(t)=2\pi\left(\frac{4}{3}t^3+2t^2+t\right)+\pi$

$V(3)=2\pi\left(\frac{4}{3}(3)^3+2(3)^2+3\right)+\pi=2\pi(57)+\pi=115\pi\ \text{cm}^3$

5 a $v(t)=20-5t$

$a(t)=v'(t)=-5\ \text{ms}^{-2}$

b $s(t)=\int v(t)\text{d}t=\int(20-5t)dt=20t-\frac{5}{2}t^2+C$

$5=20(0)-\frac{5}{2}(0)^2+C$

$C=5$

$s(t)=\int v(t)dt=5+20t-\frac{5}{2}t^2$

Exercise 9D

1 $\int\frac{2}{x}dx=2\int\frac{1}{x}dx=2\ln x+C,\ x>0$

2 $\int 3e^x dx=3\int e^x dx=3e^x+C$

3 $\int\frac{1}{4t}dt=\frac{1}{4}\int\frac{1}{t}dt=\frac{1}{4}\ln t+C,\ t>0$

4 $\int e^{\ln x}dx=\int xdx=\frac{1}{2}x^2+C$

5 $\int(2x+3)^2dx=\int(4x^2+12x+9)dx=4\left(\frac{1}{3}x^3\right)$

$+12\left(\frac{1}{2}x^2\right)+9x+C=\frac{4}{3}x^3+6x^2$

$+9x+C$

6 $\int\frac{2x^3+6x^2+5}{x}dx=\int\left(2x^2+6x+\frac{5}{x}\right)dx$

$=2\left(\frac{1}{3}x^3\right)+6\left(\frac{1}{2}x^2\right)+5\ln x+C$

$=\frac{2}{3}x^3+3x^2+5\ln x+C,\ x>0$

7 $\int\ln e^{u^2}du=\int u^2du=\frac{1}{3}u^3+C$

8 $\int(x-1)^3dx=\int(x^3-3x^2+3x-1)dx$

$=\frac{1}{4}x^4-3\left(\frac{1}{3}x^3\right)+3\left(\frac{1}{2}x^2\right)-x+C$

$=\frac{1}{4}x^4-x^3+\frac{3}{2}x^2-x+C$

9 $\int\frac{e^x+1}{2}dx=\frac{1}{2}\int(e^x+1)dx=\frac{1}{2}(e^x+x)+C$

10 $\int\frac{x^2+x+1}{\sqrt{x}}dx=\int\left(x^{\frac{3}{2}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$

$=\frac{1}{\frac{5}{2}}x^{\frac{5}{2}}+\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}+\frac{1}{\frac{1}{2}}x^{\frac{1}{2}}+C$

$=\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$

Exercise 9E

1 $\int(2x+5)^2dx=\frac{1}{2}\left(\frac{1}{3}(2x+5)^3\right)+C=\frac{1}{6}(2x+5)^3+C$

2 $\int(-3x+5)^3dx=-\frac{1}{3}\left(\frac{1}{4}(-3x+5)^4\right)+C$

$=-\frac{1}{12}(-3x+5)^4+C$

3 $\int e^{\frac{1}{2}x-3}dx=\frac{1}{\frac{1}{2}}e^{\frac{1}{2}x-3}+C=2e^{\frac{1}{2}x-3}+C$

4 $\int \frac{1}{5x+4}dx = \frac{1}{5}(\ln(5x+4))+C$

$= \frac{1}{5}\ln(5x+4)+C, \quad x > -\frac{4}{5}$

5 $\int \frac{3}{7-2x}dx = 3\int \frac{1}{7-2x}dx = 3\left(\frac{1}{-2}\ln(7-2x)\right)+C$

$= -\frac{3}{2}\ln(7-2x)+C, \quad x < \frac{7}{2}$

6 $\int 4e^{2x+1}dx = 4\int e^{2x+1}dx = 4\left(\frac{1}{2}e^{2x+1}\right)+C = 2e^{2x+1}+C$

7 $\int 6(4x-3)^7dx = 6\int (4x-3)^7dx$

$= 6\left(\frac{1}{4}\left(\frac{1}{8}(4x-3)^8\right)\right)+C$

$= \frac{3}{16}(4x-3)^8+C$

8 $\int (7x+2)^{\frac{1}{2}}dx = \frac{1}{7}\left(\frac{1}{\frac{3}{2}}(7x+2)^{\frac{3}{2}}\right)+C$

$= \frac{1}{7}\left(\frac{2}{3}(7x+2)^{\frac{3}{2}}\right)+C$

$= \frac{2}{21}(7x+2)^{\frac{3}{2}}+C$

9 $\int \left(e^{4x}+\frac{4}{3x-5}\right)dx = \int e^{4x}dx + 4\int \frac{1}{3x-5}dx$

$= \frac{1}{4}e^{4x}+4\left(\frac{1}{3}\ln(3x-5)\right)+C$

$= \frac{1}{4}e^{4x}+\frac{4}{3}\ln(3x-5)+C, \quad x > \frac{5}{3}$

10 $\int \frac{2}{3(4x-5)^3}dx = \frac{2}{3}\int (4x-5)^{-3}dx$

$= \frac{2}{3}\left(\frac{1}{4}\left(\frac{1}{-2}(4x-5)^{-2}\right)\right)+C$

$= -\frac{1}{12(4x-5)^2}+C$

11 a $f(x)=(4x+5)^3; f'(x)=3(4x+5)^2(4)=12(4x+5)^2$

b $\int f(x)dx = \int (4x+5)^3dx = \frac{1}{4}\left(\frac{1}{4}(4x+5)^4\right)+C$

$= \frac{1}{16}(4x+5)^4+C$

12 $v(t)=e^{-3t}+6t; \quad s(0)=4$

$s(t) = \int (e^{-3t}+6t)dt = \frac{1}{-3}e^{-3t}+6\left(\frac{1}{2}t^2\right)+C = -\frac{1}{3}e^{-3t}+3t^2+C$

$4 = -\frac{1}{3}e^{-3(0)}+3(0)^2+C \Rightarrow 4 = -\frac{1}{3}+C \Rightarrow C = \frac{13}{3}$

$s(t) = -\frac{1}{3}e^{-3t}+3t^2+\frac{13}{3}$

Exercise 9F

1 $\int (2x^2+5)^2(4x)dx \Rightarrow u = 2x^2+5; \frac{du}{dx} = 4x$

$\int (2x^2+5)^2(4x)dx = \int u^2\left(\frac{du}{dx}\right)dx = \int u^2du$

$= \frac{1}{3}u^3+C = \frac{1}{3}(2x^2+5)^3+C$

2 $\int \frac{3x^2+2}{x^3+2x}dx \Rightarrow u = x^3+2x; \frac{du}{dx} = 3x^2+2$

$\int \frac{3x^2+2}{x^3+2x}dx = \int \frac{\frac{du}{dx}}{u}dx = \int \frac{1}{u}du = \ln u + C$

$= \ln(x^3+2x)+C, \; x^3+2x>0$

3 $\int (6x+5)\sqrt{3x^2+5x}\,dx \Rightarrow u = 3x^2+5x; \frac{du}{dx} = 6x+5$

$\int (6x+5)\sqrt{3x^2+5x}\,dx = \int \left(\frac{du}{dx}\right)u^{\frac{1}{2}}dx = \int u^{\frac{1}{2}}du$

$= \frac{2}{3}u^{\frac{3}{2}}+C = \frac{2}{3}\left(3x^2+5x\right)^{\frac{3}{2}}+C$

4 $\int 4x^3e^{x^4}dx \Rightarrow u = x^4; \frac{du}{dx} = 4x^3$

$\int 4x^3e^{x^4}dx = \int \left(\frac{du}{dx}\right)e^udx = \int e^udu = e^u+C = e^{x^4}+C$

5 $\int \frac{2x+3}{(x^2+3x+1)^2}dx \Rightarrow u = x^2+3x+1; \frac{du}{dx} = 2x+3$

$\int \frac{2x+3}{(x^2+3x+1)^2}dx = \int \frac{\frac{du}{dx}}{u^2}dx = \int u^{-2}du = \frac{1}{-1}u^{-1}+C$

$= -\frac{1}{u}+C = -\frac{1}{x^2+3x+1}+C$

6 $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx \Rightarrow u = \sqrt{x} = x^{\frac{1}{2}}; \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = \int \left(\frac{1}{2\sqrt{x}}(e^{\sqrt{x}})\right)dx = \int \left(\frac{du}{dx}\right)e^udx = \int e^udu$

$= e^u+C = e^{\sqrt{x}}+C$

7 $\int x^2(2x^3+5)^4dx \Rightarrow u = 2x^3+5; \frac{du}{dx} = 6x^2; \frac{1}{6}\left(\frac{du}{dx}\right) = x^2$

$\int x^2(2x^3+5)^4dx = \int \frac{1}{6}\left(\frac{du}{dx}\right)u^4dx = \frac{1}{6}\int u^4du$

$= \frac{1}{6}\left(\frac{1}{5}u^5\right)+C = \frac{1}{30}(2x^3+5)^5+C$

8 $\int \frac{2x+1}{\sqrt[4]{x^2+x}}dx \Rightarrow u = x^2+x; \frac{du}{dx} = 2x+1$

$\int \frac{2x+1}{\sqrt[4]{x^2+x}}dx = \int \frac{\frac{du}{dx}}{u^{\frac{1}{4}}}dx = \int u^{-\frac{1}{4}}du = \frac{4}{3}u^{\frac{3}{4}}+C = \frac{4}{3}(x^2+x)^{\frac{3}{4}}+C$

9 $\int (8x^3-4x)(x^4-x^2)^3dx \Rightarrow u = x^4-x^2; \frac{du}{dx} = 4x^3-2x$

$\int (8x^3-4x)(x^4-x^2)^3dx = \int 2(4x^3-2x)(x^4-x^2)^3dx$

$= 2\int \left(\frac{du}{dx}\right)u^3dx = 2\left(\frac{1}{4}u^4\right)+C$

$= \frac{1}{2}(x^4-x^2)^4+C$

10 $\int \frac{4-3x^2}{x^3-4x}dx \Rightarrow u = x^3-4x;\ \frac{du}{dx} = 3x^2-4;\ -\frac{du}{dx} = 4-3x^2$

$$\int \frac{4-3x^2}{x^3-4x}dx = \int \frac{\left(-\frac{du}{dx}\right)}{u}dx = -\int \frac{1}{u}du = -\ln u + C$$

$$= -\ln\left(x^3-4x\right)+C;\ x^3-4x>0$$

11 $f'(x) = \frac{8x}{4x^2+1};\ f(0) = 4$

$\int \frac{8x}{4x^2+1}dx \Rightarrow u = 4x^2+1;\ \frac{du}{dx} = 8x$

$\int \frac{8x}{4x^2+1}dx = \int \frac{\frac{du}{dx}}{u}dx = \int \frac{1}{u}du = \ln(4x^2+1)+C$

$\ln(4(0)^2+1)+C = 4 \Rightarrow \ln 1 + C = 4 \Rightarrow C = 4$

$f(x) = \ln(4x^2+1)+4$

12 $f'(x) = 3x^2e^{x^3}\ (1, 5e)$

$f(x) = \int 3x^2e^{x^3}dx \Rightarrow u = x^3;\ \frac{du}{dx} = 3x^2$

$f(x) = \int 3x^2e^{x^3}dx = \int \left(\frac{du}{dx}\right)e^u dx = \int e^u du$

$= e^u + C = e^{x^3} + C$

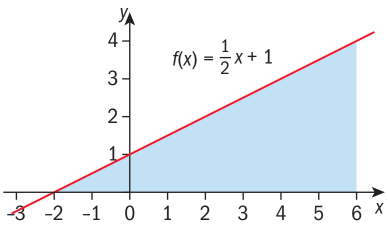
$e^{1^3} + C = 5e \Rightarrow C = 4e$

$f(x) = e^{x^3} + 4e$

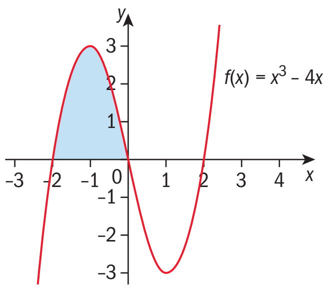
Exercise 9G

1 Use the area formula for a triangle to verify your answer.

$\int_{-2}^{6}\left(\frac{1}{2}x+1\right)dx = 16;\ \frac{1}{2}(8)(4) = 16$

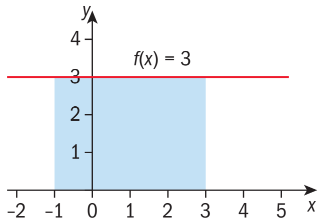


2 $\int_{-2}^{0}\left(x^3-4x\right)dx = 4$; no area formula



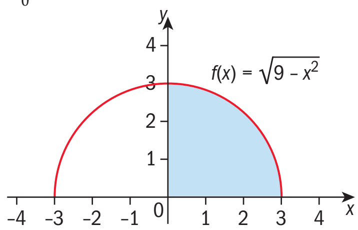
3 Use the area formula for a rectangle to verify your answer.

$\int_{-1}^{3} 3dx = 12;\ 4(3) = 12$

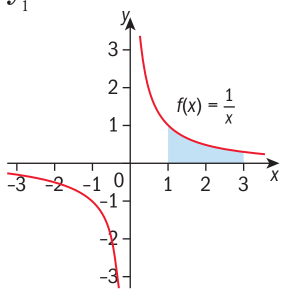


4 Find one-fourth the area of a circle to verify your answer.

$\int_{0}^{3}\sqrt{9-x^2}dx \approx 7.07;\ \frac{1}{4}\pi(3^2) \approx 7.07$

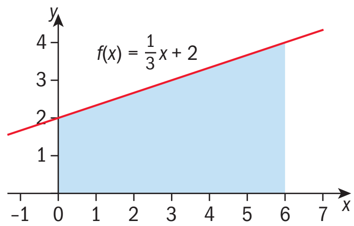


5 $\int_{1}^{3}\frac{1}{x}dx \approx 1.10$; no area formula



6 Use the area formula for a trapezoid to verify your answer.

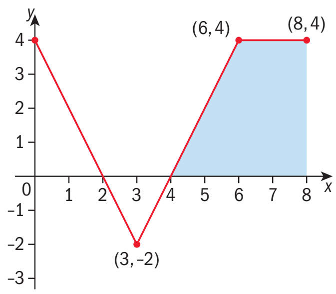
$\int_{0}^{6}\left(\frac{1}{3}x+2\right)dx = 18;\ \frac{1}{2}(2+4)(6) = 18$



Exercise 9H

1 Use the area formula for a trapezoid.

$\int_{4}^{8} f(x)dx = \frac{1}{2}(4+2)(4) = 12$



2 Find the area of the triangle above the x-axis, subtract the area of the triangle below the x-axis and then add the area of the trapezoid.

$$\int_0^8 f(x)dx = \tfrac{1}{2}(2)(4) - \tfrac{1}{2}(2)(2) + \tfrac{1}{2}(4+2)(4)$$
$$= 4 - 2 + 12 = 14$$

In questions 3–10 we are given that

$$\int_1^6 f(x)dx = -3,\ \int_1^{10} f(x)dx = 8,\ \int_1^6 g(x)dx = 4, \text{ and}$$
$$\int_6^{10} g(x)dx = 8$$

3 $$\int_1^6 \left(2f(x) + \tfrac{1}{2}g(x)\right)dx =$$
$$2\int_1^6 f(x)dx + \tfrac{1}{2}\int_1^6 g(x)dx = 2(-3) + \tfrac{1}{2}(4) = -4$$

4 $$\int_{10}^6 g(x)dx = -\int_6^{10} g(x)dx = -8$$

5 $$\int_1^{10} g(x)dx = \int_1^6 g(x)dx + \int_6^{10} g(x)dx = 4 + 8 = 12$$

6 $$\int_a^a f(x)dx = 0 \Rightarrow \int_{10}^{10} f(x)dx = 0$$

7 $$\int_1^6 f(x)dx + \int_6^{10} f(x)dx = \int_1^{10} f(x)dx \Rightarrow -3 + \int_6^{10} f(x)dx$$
$$= 8 \Rightarrow \int_6^{10} f(x)dx = 11$$

8 We know that $\int_1^6 f(x)dx = -3$. The graph of $y = f(x-4)$ is the graph of $y = f(x)$ translated 4 units to the right and the limits of integration $x = 1$ and $x = 6$ are translated to $x = 5$ and $x = 10$, so $\int_5^{10} f(x-4)dx = \int_1^6 f(x)dx = -3$

9 The graph of $y = g(x) + 3$ is the graph of $y = g(x)$ translated up 3. This adds a rectangular region of length 4 and height 3 to the area under the curve, so $\int_6^{10} (g(x) + 3)dx = \int_6^{10} g(x)dx + \int_6^{10} 3dx = 8 + 4(3) = 20$

10 We know that $\int_1^6 g(x)dx = 4$. The graph of $y = g(x + 2)$ is the graph of $y = g(x)$ translated 2 units to the left and the limits of integration $x = 1$ and $x = 6$ are translated to $x = -1$ and $x = 4$, so

$$\int_{-1}^4 3g(x+2)dx = 3\int_{-1}^4 g(x+2)dx = 3\int_1^6 g(x)dx = 3(4) = 12$$

11 Given that $\int_0^2 h(x)dx = -2$ and $\int_2^5 h(x)dx = 6$

a $$\int_0^5 h(x)dx = \int_0^2 h(x)dx + \int_2^5 h(x)dx = -2 + 6 = 4$$

b $$\int_2^5 (h(x) + 2)dx = \int_2^5 h(x)dx + \int_2^5 2dx = 6 + 3(2) = 12$$

12 a We know that $\int_0^4 f(x)dx = 16$ and so

$$\int_0^4 \tfrac{1}{4}f(x)dx = \tfrac{1}{4}\int_0^4 f(x)dx = \tfrac{1}{4}(16) = 4$$

b i We know that $\int_0^4 f(x)dx = 16$ and since $y = f(x-3)$ is the translation of the graph of $y = f(x)$ to the right 3 units and the limits of integration $x = 0$ and $x = 4$ are translated to $x = 3$ and $x = 7$. So if $\int_a^b f(x-3)dx = 16$, we can deduce that $a = 3$ and $b = 7$

ii $$\int_0^4 \left(f(x)+k\right)dx = \int_0^4 (f(x))dx + \int_0^4 kdx = 16 + 4(k)$$

and $\int_0^4 \left(f(x)+k\right)dx = 28 \Rightarrow 16 + 4k = 28 \Rightarrow k = 3$

Exercise 9I

1 $$\int_0^1 2xdx = 2\int_0^1 x\,dx = 2\left[\frac{1}{2}x^2\right]_0^1 = 2\left[\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2\right] = 1$$

2 $$\int_{-1}^1 (u^2 - 2)du = \left[\tfrac{1}{3}u^3 - 2u\right]_{-1}^1 = \left[\tfrac{1}{3}(1)^3 - 2(1)\right]$$
$$-\left[\tfrac{1}{3}(-1)^3 - 2(-1)\right] = -\frac{5}{3} - \frac{5}{3} = -\frac{10}{3}$$

3 $$\int_{-1}^2 \left(\frac{3}{x^2} - 1\right)dx = \int_{-1}^2 \left(3x^{-2} - 1\right)dx = \left[-3x^{-1} - x\right]_1^2$$
$$= \left[-\tfrac{3}{2} - 2\right] - \left[-3 - 1\right] = -\tfrac{7}{2} + 4 = \tfrac{1}{2}$$

4 $$\int_0^8 \left(x^{\frac{1}{3}} - x^{\frac{2}{3}}\right)dx = \left[\frac{3}{4}x^{\frac{4}{3}} - \frac{3}{5}x^{\frac{5}{3}}\right]_0^8 = \left[\frac{3}{4}(8)^{\frac{4}{3}} - \frac{3}{5}(8)^{\frac{5}{3}}\right]$$
$$= \left[\frac{3}{4}(0)^{\frac{4}{3}} - \frac{3}{5}(0)^{\frac{5}{3}}\right] = \left[12 - \frac{96}{5}\right] - 0 = -\frac{36}{5}$$

5 $$\int_0^8 4e^x dx = 4\int_0^8 e^x dx = 4\left[e^x\right]_0^3 = 4\left(e^3 - e^0\right) = 4(e^3 - 1)$$

6 $$\int_e^{e^2} \frac{1}{x}dx = [\ln x]_e^{e^2} = \ln e^2 - \ln e = 2 - 1 = 1$$

7 $$\int_0^1 (t+3)(t+1)dt = \int_0^1 (t^2 + 4t + 3)dt$$
$$= \left[\tfrac{1}{3}t^3 + 2t^2 + 3t\right]_0^1$$
$$= \left[\tfrac{1}{3}(1)^3 + 2(1)^2 + 3(1)\right]$$
$$-\left[\tfrac{1}{3}(0)^3 + 2(0)^2 + 3(0)\right] = \frac{16}{3}$$

8 $$\int_4^9 \frac{2\sqrt{x} + 3}{\sqrt{x}}dx = \int_4^9 \left(2 + 3x^{-\frac{1}{2}}\right)dx = \left[2x + 6x^{\frac{1}{2}}\right]_4^9$$
$$= \left[2(9) + 6(9)^{\frac{1}{2}}\right] - \left[2(4) + 6(4)^{\frac{1}{2}}\right] = 36 - 20 = 16$$

9 **a** Given that $\int_0^2 f(x)dx = 8$,

$$\int_0^2 3f(x)dx = 3\int_0^2 f(x)dx = 3(8) = 24$$

b Given that $\int_0^2 f(x)dx = 8$,

$$\int_0^2 \left(f(x)+x^2\right)dx = \int_0^2 f(x)dx + \int_0^2 x^2dx = 8+\left[\tfrac{1}{3}x^3\right]_0^2$$

$$= 8+\left[\tfrac{1}{3}(2)^3 - \tfrac{1}{3}(0)^3\right] = \frac{32}{3}$$

10 $\int_2^k \frac{1}{x}dx = \ln 6 \Rightarrow \left[\ln x\right]_2^k = \ln 6 \Rightarrow$

$\ln k - \ln 2 = \ln 6 \Rightarrow \ln\frac{k}{2} = \ln 6 \Rightarrow \frac{k}{2} = 6 \Rightarrow k = 12$

Exercise 9J

1 $\int_{-1}^1 \frac{1}{t+2}dt = \left[\frac{1}{1}\ln(t+2)\right]_{-1}^1$

$= \left[\ln(1+2)\right] - \left[\ln(-1+2)\right] = \ln(3) - \ln(1)$

$= \ln(3) - 0 = \ln 3$

2 $\int_3^4 e^{-x+1}\,dx = \left[\frac{1}{-1}e^{-x+1}\right]_3^4 = \left[-e^{-4+1}\right] - \left[-e^{-3+1}\right]$

$= -e^{-3} + e^{-2} = \frac{1}{e^2} - \frac{1}{e^3}$

3 $\int_{-1}^2 (-2x+1)^3dx = \left[\frac{1}{-2}\left(\frac{1}{4}(-2x+1)^4\right)\right]_{-1}^2$

$= \left[-\frac{1}{8}(-2(2)+1)^4\right] - \left[-\frac{1}{8}(-2(-1)+1)^4\right]$

$= -\frac{81}{8} + \frac{81}{8} = 0$

4 $\int_{-1}^1 (e^x + e^{-x})dx = \left[e^x + \frac{1}{-1}e^{-x}\right]_{-1}^1 = \left[e^1 - e^{-1}\right]$

$-\left[e^{-1} - e^{-1(-1)}\right] = \left[e - \frac{1}{e}\right] - \left[\frac{1}{e} - e\right] = 2\left(e - \frac{1}{e}\right)$

5 $\int_{-1}^1 \sqrt{6x+4}\,dx = \int_{-1}^1 (6x+4)^{\frac{1}{2}}\,dx$

$= \left[\frac{1}{6}\left(\frac{2}{3}(6x+4)^{\frac{3}{2}}\right)\right]_0^2$

$= \left[\frac{1}{9}(6(2)+4)^{\frac{3}{2}}\right] - \left[\frac{1}{9}(6(0)+4)^{\frac{3}{2}}\right]$

$= \frac{64}{9} - \frac{8}{9} = \frac{56}{9}$

6 $\int_1^2 (x^2+x)^3(2x+1)dx \Rightarrow u = x^2 + x;\ \frac{du}{dx} = 2x+1;$

when $x=1$, $u=1^2+1=2$ when $x=2$, $u=2^2+2=6$

$\int_1^2 (x^2+x)^3(2x+1)dx = \int_{x=1}^{x=2} u^3\left(\frac{du}{dx}\right)dx = \int_{u=2}^{u=6} u^3du$

$= \left[\frac{1}{4}u^4\right]_2^6 = \left[\frac{1}{4}(6)^4\right] - \left[\frac{1}{4}(2)^4\right] = 324 - 4 = 320$

7 $\int_3^4 \frac{8t-6}{2t^2-3t-2}dt = \int_3^4 \frac{2(4t-3)}{2t^2-3t-2}dt$

$= 2\int_3^4 \frac{4t-3}{2t^2-3t-2}dt \Rightarrow u = 2t^2 - 3t - 2;\ \frac{du}{dt} = 4t - 3$

when $t=3$, $u=2(3)^2-3(3)-2=7$ and when $t=4$, $u=2(4)^2-3(4)-2=18$

$$2\int_3^4 \frac{4t-3}{2t^2-3t-2}dt = 2\int_{t=3}^{t=4} \frac{\left(\frac{du}{dt}\right)}{u}dt = 2\int_{u=7}^{u=18} \frac{1}{u}du$$

$= 2\left[\ln u\right]_7^{18} = 2\left[\ln 18 - \ln 7\right] = 2\ln\frac{18}{7}$

8 $\int_0^1 4xe^{x^2+3}dx = 2\int_0^1 2xe^{x^2+3}dx \Rightarrow u = x^2+3;\ \frac{du}{dx} = 2x;$

when $x=0$, $u=0^2+3=3$ and when $x=1$, $u=1^2+3=4$

$2\int_0^1 2xe^{x^2+3}dx = 2\int_{x=0}^{x=1}\left(\frac{du}{dx}\right)e^u dx = 2\int_{u=3}^{u=4} e^u du$

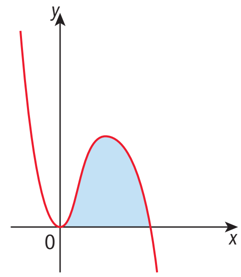
$= 2\left[e^u\right]_3^4 = 2(e^4 - e^3)$

9 **a** $f(x) = -2x^2(x-2)$.

$f(x)$ cuts the x-axis when $f(x) = 0$, so

$-2x^2(x-2) = 0 \Rightarrow x = 0, 2$

Area $= \int_0^2 -2x^2(x-2)dx$



b Area $= \int_0^2 -2x^2(x-2)dx = -2\int_0^2 (x^3-2x^2)dx$

$= -2\left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_0^2$

$= -2\left[\left(\frac{1}{4}(2)^4 - \frac{2}{3}(2)^3\right) - \left(\frac{1}{4}(0)^4 - \frac{2}{3}(0)^3\right)\right]$

$= -2\left[\left(4 - \frac{16}{3}\right) - 0\right] = \frac{8}{3}$

10 $\int_2^k \frac{1}{x-1}dx = \ln 4$

$\int_2^k \frac{1}{x-1}dx = \frac{1}{1}\left[\ln(x-1)\right]_2^k = \left[\ln(k-1)\right] - \left[\ln(2-1)\right]$

$= \left[\ln(k-1)\right] - \left[\ln(1)\right] = \ln(k-1)$

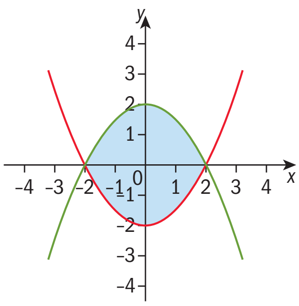
$\ln(k-1) = \ln 4 \Rightarrow k - 1 = 4 \Rightarrow k = 5$

Exercise 9K

1 $y = -\frac{1}{2}x^2 + 2$ and $y = \frac{1}{2}x^2 - 2$

Find the x-coordinates of the intersection points.

$-\frac{1}{2}x^2 + 2 = \frac{1}{2}x^2 - 2 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$



$$\int_{-2}^{2}\left(\left(-\frac{1}{2}x^2+2\right)-\left(\frac{1}{2}x^2-2\right)\right)dx=\int_{-2}^{2}\left(-x^2+4\right)dx$$

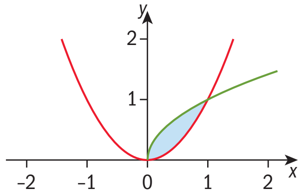
$$=\left[-\frac{1}{3}x^3+4x\right]_{-2}^{2}$$

$$=\left[-\frac{1}{3}(2)^3+4(2)\right]$$

$$-\left[-\frac{1}{3}(-2)^3+4(-2)\right]$$

$$=\frac{16}{3}-\left(-\frac{16}{3}\right)=\frac{32}{3}$$

2 $f(x)=x^2$ and $g(x)=\sqrt{x}$



Find the x-coordinates of the intersection points.

$$x^2=\sqrt{x}\Rightarrow x^4=x\Rightarrow x^4-x=0\Rightarrow x(x^3-1)=0\Rightarrow x=0,1$$

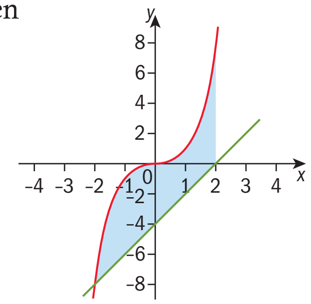
$$\int_0^1\left(\sqrt{x}-x^2\right)dx=\int_0^1\left(x^{\frac{1}{2}}-x^2\right)dx=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{3}x^3\right]_0^1$$

$$=\left[\frac{2}{3}(1)^{\frac{3}{2}}-\frac{1}{3}(1)^3\right]-\left[\frac{2}{3}(0)^{\frac{3}{2}}-\frac{1}{3}(0)^3\right]=\frac{1}{3}$$

3 $y = 2x - 4$, $y = x^3$ between $x = -2 =$ and $x = 2$

Using GDC, We see $2x - 4 = x^3$ has no roots for $-2 < x < 2$.

At $x = 0$, $x^3 > 2x - 4$, so we can use $\int_{-2}^{2} x^3 - (2x - 4)dx$ to find bounded area.



$$\int_{-2}^{2}(x^3-(2x-4))\,dx=\int_{-2}^{2}(x^3-2x+4)\,dx$$

$$=\left[\frac{1}{4}x^4-x^2+4x\right]_{-2}^{2}$$

$$=\left[\frac{1}{4}(2)^4-(2)^2+4(2)\right]-$$

$$\left[\frac{1}{4}(-2)^4-(-2)^2+4(-2)\right]$$

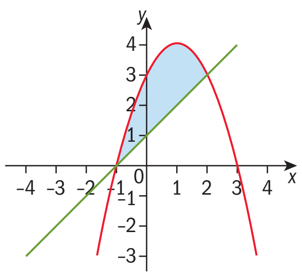
$$=8-(-8)=16$$

4 $g(x)=x+1$ and $h(x)=3+2x-x^2$

Find the x-coordinates of the intersection points.

$$x+1=3+2x-x^2\Rightarrow x^2-x-2=0\Rightarrow$$

$$(x+1)(x-2)=0\Rightarrow x=-1,2$$



$$\int_{-1}^{2}\left((3+2x-x^2)-(x+1)\right)dx=\int_{-1}^{2}(2+x-x^2)dx$$

$$=\left[2x+\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-1}^{2}$$

$$=\left[2(2)+\frac{1}{2}(2)^2-\frac{1}{3}(2)^3\right]-\left[2(-1)+\frac{1}{2}(-1)^2-\frac{1}{3}(-1)^3\right]$$

$$=\left(6-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)$$

$$=\frac{10}{3}-\left(-\frac{7}{6}\right)=\frac{27}{6}=\frac{9}{2}$$

5 **a** $f(x)=x^4-x^2$

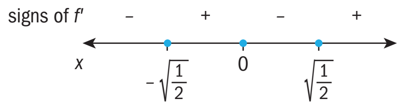
$x^4-x^2=0\Rightarrow x^2(x^2-1)=0\Rightarrow x=0,-1,1$

The x-intercepts are (0, 0), (–1, 0) and (1, 0).

b **i** $f'(x)=4x^3-2x$.

ii $f'(x)=4x^3-2x$

$$4x^3-2x=0\Rightarrow 2x\left(2x^2-1\right)=0\Rightarrow x=0,-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}}$$

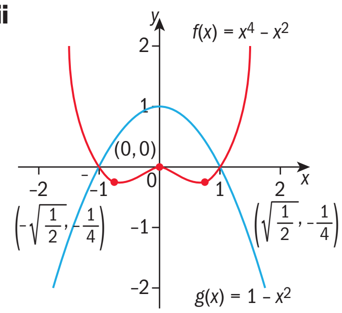


$$f(0)=0;\ f\left(-\sqrt{\frac{1}{2}}\right)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4};\ f\left(\sqrt{\frac{1}{2}}\right)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$$

Relative minimum points: $\left(-\sqrt{\frac{1}{2}},-\frac{1}{4}\right),\left(\sqrt{\frac{1}{2}},-\frac{1}{4}\right)$

Relative maximum points: (0, 0)

c **i and ii**



d $\int_{-1}^{1}\left((1-x^2)-(x^4-x^2)\right)dx=\int_{-1}^{1}(1-x^4)dx$

$$=\left[x-\frac{1}{5}x^5\right]_{-1}^{1}$$

$$=\left[1-\frac{1}{5}(1)^5\right]$$

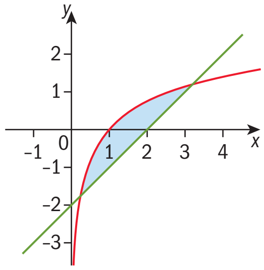
$$-\left[-1-\frac{1}{5}(-1)^5\right]$$

$$=\frac{4}{5}-\left(-\frac{4}{5}\right)=\frac{8}{5}$$

6 $y = \ln x$ and $y = x - 2$

Use a GDC to help sketch the graph, find x-coordinates of the points of intersection and to evaluate the definite integral.

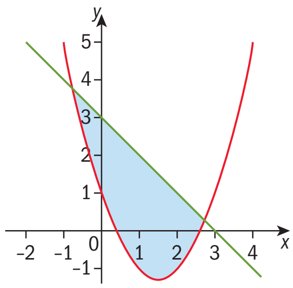
$$\int_{0.1586}^{3.146}\left(\ln(x)-(x-2)\right)dx\approx 1.95$$



7 $f(x) = x^2 - 3x + 1$ and $g(x) = -x + 3$

Use a GDC to help sketch the graph, find x-coordinates of the points of intersection and to evaluate the definite integral.

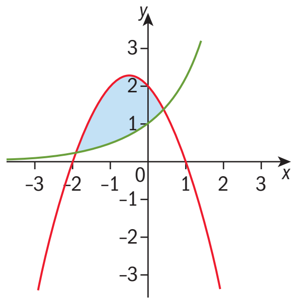
$$\int_{-0.7321}^{2.732} \left((-x+3) - (x^2 - 3x + 1)\right) dx \approx 6.93$$



8 $f(x) = e^x$ and $h(x) = 2 - x - x^2$

Use a GDC to help sketch the graph, find x-coordinates of the points of intersection and to evaluate the definite integral.

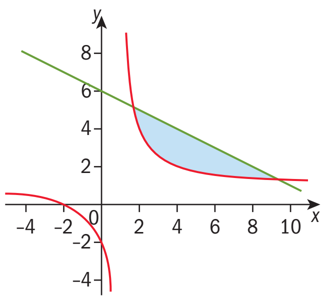
$$\int_{-1.952}^{0.3841} \left((2 - x - x^2) - e^x\right) dx \approx 2.68$$



9 $y = \frac{x+2}{x-1}$ and $y = -\frac{1}{2}x + 6$

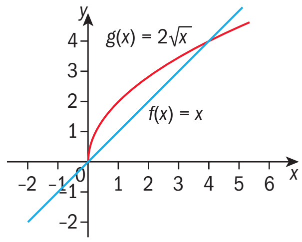
Use a GDC to help sketch the graph, find x-coordinates of the points of intersection and to evaluate the definite integral.

$$\int_{1.725}^{9.275} \left(\left(-\frac{1}{2}x + 6 \right) - \frac{x+2}{x-1} \right) dx \approx 9.68$$



10 a $f(x) = x$ and $g(x) = 2\sqrt{x}$

Use a GDC to help sketch the graphs.



b i $\int_0^4 (2\sqrt{x} - x)dx$

ii Use a GDC to evaluate:

$\int_0^4 \left(2\sqrt{x} - x\right)dx \approx 2.67.$

c i $\int_0^k \left(2\sqrt{x} - x\right)dx$

ii
$$\int_0^k \left(2\sqrt{x} - x\right)dx = \int_0^k \left(2x^{\frac{1}{2}} - x\right)dx$$
$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^k$$
$$= \left[\frac{4}{3}k^{\frac{3}{2}} - \frac{1}{2}k^2\right] - \left[\frac{4}{3}(0)^{\frac{3}{2}} - \frac{1}{2}(0)^2\right]$$
$$= \frac{4}{3}k^{\frac{3}{2}} - \frac{1}{2}k^2$$

Use a GDC to solve: $\frac{4}{3}k^{\frac{3}{2}} - \frac{1}{2}k^2 \approx \frac{1}{2}$ (2.66667)

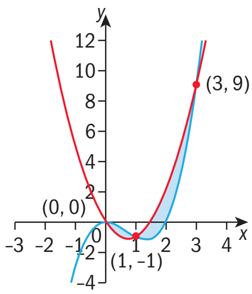
$\approx 1.333 \Rightarrow k \approx 1.51$ or 6.22

Since $0 < k < 4$, we know $k \approx 1.51$

Exercise 9L

1 $y = x^3 - 2x^2$ and $y = 2x^2 - 3x$

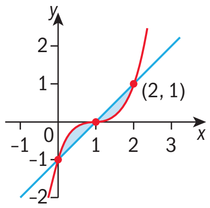
Use a GDC to help sketch the graph and find the points of intersection and evaluate the definite integral.



$$\int_0^1 \left((x^3 - 2x^2) - (2x^2 - 3x)\right)dx +$$
$$\int_1^3 \left((2x^2 - 3x) - (x^3 - 2x^2)\right)dx \approx 3.08$$

2 $f(x) = (x-1)^3$ and $g(x) = x - 1$

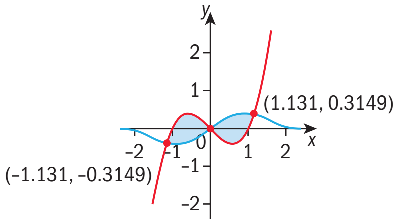
Use a GDC to help sketch the graph and find the points of intersection and evaluate the definite integral.



$$\int_0^1 \left((x-1)^3-(x-1)\right)dx+$$
$$\int_1^2 \left((x-1)-(x-1)^3\right)dx = 0.5$$

3 $f(x) = xe^{-x^2}$ and $g(x) = x^3 - x$

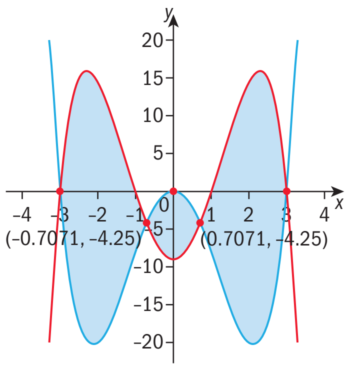
Use a GDC to help sketch the graph and find the points of intersection and evaluate the definite integral.



$$\int_{-1.131}^0 \left((x^3-x)-(xe^{-x^2})\right) dx+$$
$$\int_0^{1.131} \left((xe^{-x^2})-(x^3-x)\right) dx = 1.18$$

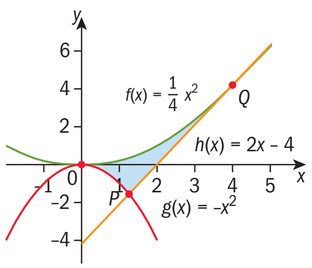
4 $g(x) = -x^4 + 10x^2 - 9$ and $h(x) = x^4 - 9x^2$

Use a GDC to help sketch the graph and find the points of intersection and evaluate the definite integral.



$$\int_{-3}^{-0.7071} \left((-x^4+10x^2-9)-(x^4-9x^2)\right)dx$$
$$+\int_{-0.7071}^{0.7071} \left((x^4-9x^2)-(-x^4+10x^2-9)\right)dx$$
$$+\int_{0.7071}^{3} \left((-x^4+10x^2-9)-(x^4-9x^2)\right)dx \approx 110$$

5



a **i** Use a GDC to find the intersection of f and h: $Q(4, 4)$

ii $f(x) = \frac{1}{4}x^2 \Rightarrow f'(x) = \frac{1}{2}x$

$$\Rightarrow m = f'(4) = 2$$

tangent to $f(x)$ at $x = 4$ is given by

$$y - 4 = 2(x-4) \Rightarrow y - 4 = 2x - 8$$
$$\Rightarrow y = 2x - 4 = h(x)$$

Thus, $h(x)$ is a tangent to $f(x)$ at the point Q.

b **i** Use a GDC to find the intersection of g and h: $P(1.236, -1.528)$

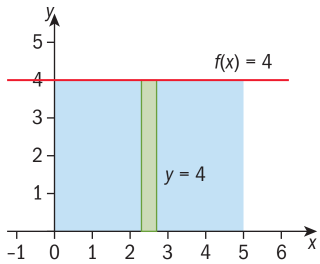
ii Area of shaded region =

$$\int_0^{1.236} f(x) - g(x)dx + \int_{1.236}^4 f(x) - h(x)dx.$$

$$\int_0^{1.236}\left(\frac{1}{4}x^2-(-x^2)\right)dx+$$
$$\int_{1.236}^{4}\left(\frac{1}{4}x^2-(2x-4)\right)dx \approx 2.55$$

Exercise 9M

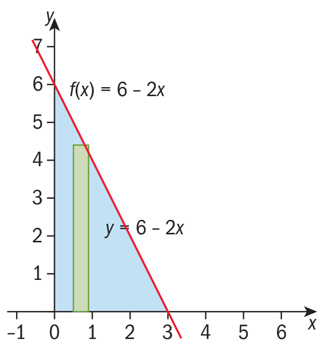
1 $f(x) = 4$ and the x-axis between $x = 0$ and $x = 5$



$$\int_0^5 \pi(4^2)dx \approx 251$$

The solid formed is a cylinder with $V = \pi(4^2)(5) \approx 251$

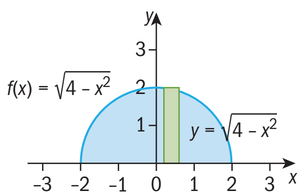
2 $f(x) = 6 - 2x$ and the x-axis between $x = 0$ and $x = 3$



$$\int_0^3 \pi(6-2x)^2 dx \approx 113$$

The solid formed is a cone with $V = \frac{1}{3}\pi(6^2)(3) \approx 113$

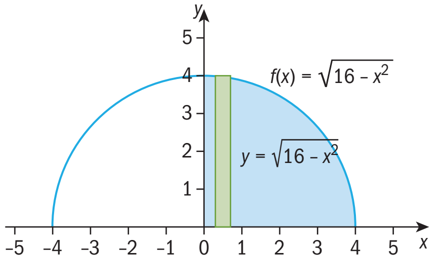
3 $f(x) = \sqrt{4 - x^2}$ and the x-axis



$$\int_{-2}^{2} \pi(\sqrt{4-x^2})^2\,dx \approx 33.5$$

The solid formed is a sphere with $V = \frac{4}{3}\pi(2^3) \approx 33.5$

4 $f(x) = \sqrt{16-x^2}$ and the x-axis between $x = 0$ and $x = 4$

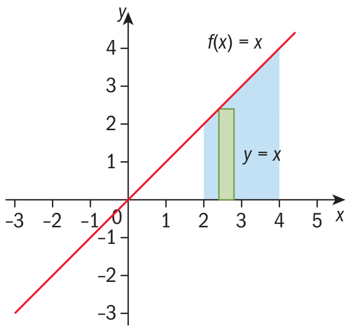


$$\int_{0}^{4} \pi(\sqrt{16-x^2})^2\,dx \approx 134$$

The volume of a half of a sphere is

$$V = \frac{1}{2}\left(\frac{4}{3}\pi(4^3)\right) \approx 134$$

5 $f(x) = x$ and the x-axis between $x = 2$ and $x = 4$



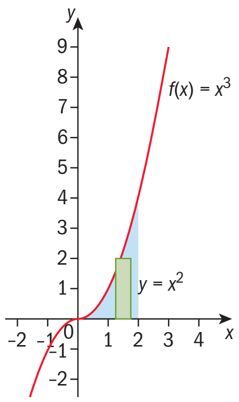
$$\int_{2}^{4} \pi(x^2)\,dx \approx 58.6$$

The solid is a frustum, a truncated cone. Find the volume of the cone formed by rotating the region under the curve from $x = 0$ to $x = 4$ minus the volume of the cone formed by rotating the region under the curve from $x = 0$ to $x = 2$:

$$V = \frac{1}{3}\pi(4^2)(4) - \frac{1}{3}\pi(2^2)(2) \approx 58.6$$

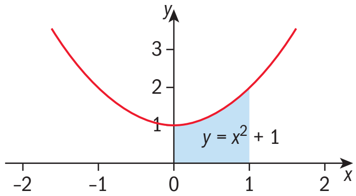
Exercise 9N

1 $f(x) = x^3$ and the x-axis between $x = 1$ and $x = 2$



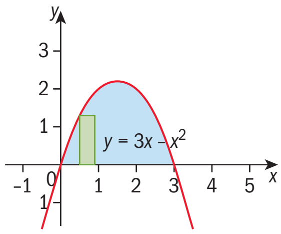
$$\int_{1}^{2} \pi(x^3)^2\,dx = \pi\int_{1}^{2} x^6\,dx = \pi\left[\frac{1}{7}x^7\right]_1^2$$
$$= \pi\left[\left(\frac{1}{7}(2)^7\right) - \left(\frac{1}{7}(1)^7\right)\right]$$
$$= \frac{127\pi}{7}$$

2 $y = x^2 + 1$ axis between $x = 0$ and $x = 1$



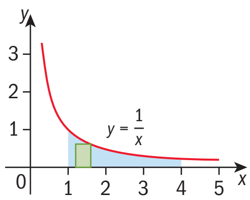
$$\int_{0}^{1} \pi(x^2+1)^2\,dx = \int_{0}^{1} \pi(x^4 + 2x^2 + 1)\,dx$$
$$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^1$$
$$= \pi\left[\frac{1}{5}(1)^5 + \frac{2}{3}(1)^3 + (1)\right]$$
$$- \pi\left[\frac{1}{5}(0)^5 + \frac{2}{3}(0)^3 + (0)\right]$$
$$= \frac{28\pi}{15}$$

3 $f(x) = 3x - x^2$ and the x-axis



$$\int_{0}^{3} \pi(3x - x^2)^2\,dx = \pi\int_{0}^{3} (9x^2 - 6x^3 + x^4)\,dx$$
$$= \pi\left[3x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5\right]_0^3$$
$$= \pi\left[3(3)^3 - \frac{3}{2}(3)^4 + \frac{1}{5}(3)^5\right]$$
$$- \pi\left[3(0)^3 - \frac{3}{2}(0)^4 + \frac{1}{5}(0)^5\right]$$
$$= \frac{81\pi}{10}$$

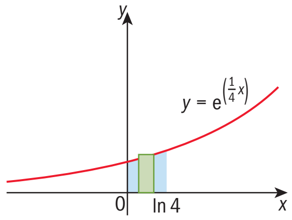
4 $y = \frac{1}{x}$ axis between $x = 1$ and $x = 4$



$$\int_1^4 \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^4 x^{-2}dx = \pi\left[-x^{-1}\right]_1^4$$

$$= \pi\left[-\frac{1}{4}-(-1)\right] = \frac{3\pi}{4}$$

5 a



a $\displaystyle\int_0^{\ln 4} \pi\left(e^{\left(\frac{1}{4}x\right)}\right)^2 dx$

b $\displaystyle\int_0^{\ln 4} \pi\left(e^{\left(\frac{1}{4}x\right)}\right)^2 dx = k\pi$

$$\pi\int_0^{\ln 4} e^{\frac{1}{2}x}\,dx = \pi\left[2e^{\frac{1}{2}x}\right]_0^{\ln 4}$$

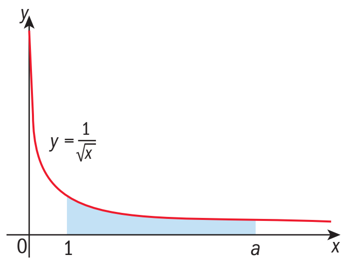
$$= \pi\left[\left(2e^{\frac{1}{2}(\ln 4)}\right)-\left(2e^{\frac{1}{2}(0)}\right)\right]$$

$$= \pi\left[\left(2e^{\ln 2}\right)-\left(2e^{0}\right)\right]$$

$$= \pi(4-2) = 2\pi$$

$$k\pi = 2\pi \Rightarrow k = 2$$

6 a



$\displaystyle\int_1^a \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx$

b $\displaystyle\int_1^a \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx = 3\pi$

$$\int_1^a \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx = \pi\int_1^a \frac{1}{x}dx = \pi\left[\ln x\right]_1^a$$

$$= \pi\left[\ln(a)-\ln(1)\right] = \pi\ln a$$

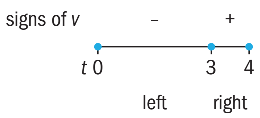
$$\pi\ln a = 3\pi \Rightarrow \ln a = 3 \Rightarrow a = e^3$$

Exercise 90

1 $s(t) = t^2 - 6t + 8;\ 0 \le t \le 4$

a $v(t) = s'(t) = 2t - 6$

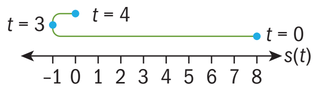
b $v(t) = 0 \Rightarrow 2t - 6 = 0 \Rightarrow t = 3$



$s(0) = 8$

$s(3) = 3^2 - 6(3) + 8 = -1$

$s(4) = 4^2 - 6(4) + 8 = 0$



c Displacement: $\int_0^4 (2t-6)dt = -8$ m

On the motion diagram we see that 0 − 8 = −8

Total distance: $\int_0^4 |2t-6|dt = 10$ m

On the motion diagram we see that

$|-1-8| + |0-(-1)| = 9 + 1 = 10$

2 $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t;\ 0 \le t \le 6$

a $v(t) = s'(t) = t^2 - 6t + 8$

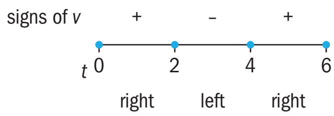
b $v(t) = 0 \Rightarrow t^2 - 6t + 8 = 0 \Rightarrow$

$(t-2)(t-4) = 0 \Rightarrow t = 2, 4$

$s(0) = 0$

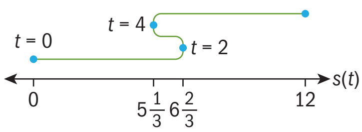
$s(2) = \frac{1}{3}(2)^3 - 3(2)^2$

$+8(2) = 6\frac{2}{3}$



$s(4) = \frac{1}{3}(4)^3 - 3(4)^2 + 8(4) = 5\frac{1}{3}$

$s(6) = \frac{1}{3}(6)^3 - 3(6)^2 + 8(6) = 12$



c Displacement: $\int_0^6 \left(t^2 - 6t + 8\right) dt = 12$ m

On the motion diagram we see that 12 − 0 = 0

Total distance: $\int_0^6 \left|t^2 - 6t + 8\right| dt \approx 14.7$ m

On the motion diagram we see that

$$\left|6\tfrac{2}{3}-0\right| + \left|5\tfrac{1}{3}-6\tfrac{2}{3}\right| = \left|12-5\tfrac{1}{3}\right|$$

$$= \frac{20}{3} + \frac{4}{3} + \frac{20}{3}$$

$$= \frac{44}{3} \approx 14.7$$

3 $s(t) = (t-2)^3;\ 0 \le t \le 4$

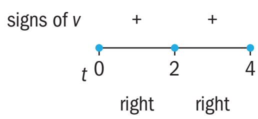
a $v(t) = s'(t) = 3(t-2)^2(1) = 3(t-2)^2$

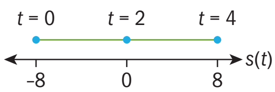
b $v(t) = 0 \Rightarrow 3(t-2)^2 = 0 \Rightarrow t = 2$

$s(0) = (0-2)^3 = -8$

$s(2) = (2-2)^3 = 0$

$s(4) = (4-2)^3 = 8$





c Displacement: $\int_0^4 3(t-2)^2 dt = 16$ m

On the motion diagram we see that

8 − (−8) = 16

Total distance: $\int_0^4 \left|3(t-2)^2\right| dt = 16$ m

On the motion diagram we see that

|8 − (−8)| = 16

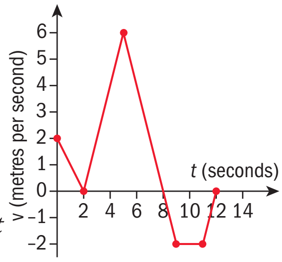
4 a Displacement: $\int_2^{12} v(t)dt$

$= \frac{1}{2}(6)(6) - \frac{1}{2}(4+2)(2)$

$= 12$ m

Total distance: $\int_2^{12} |v(t)| dt$

$= \frac{1}{2}(6)(6) + \frac{1}{2}(4+2)(2) = 24$ m



b Displacement: $\int_0^5 v(t)dt = \frac{1}{2}(2)(2) + \frac{1}{2}(3)(6)$

$= 11$ m

Total distance: $\int_0^5 |v(t)| dt = \frac{1}{2}(2)(2) + \frac{1}{2}(3)(6)$

$= 11$ m

c Displacement: $\int_0^{12} v(t)dt = \frac{1}{2}(2)(2) + \frac{1}{2}(6)(6)$

$- \frac{1}{2}(4+2)(2) = 14$ m

Total distance: $\int_0^{12} |v(t)| dt = \frac{1}{2}(2)(2) + \frac{1}{2}(6)(6)$

$+ \frac{1}{2}(4+2)(2) = 26$ m

5 a $v(t) = t^2 - 9$

$a(t) = v'(t) = 2t$

$a(1) = 2 \text{ ms}^{-2}$

b $s(t) = \int v(t)\, dt = \int \left(t^2 - 9\right) dt = \frac{1}{3}t^3 - 9t + C$

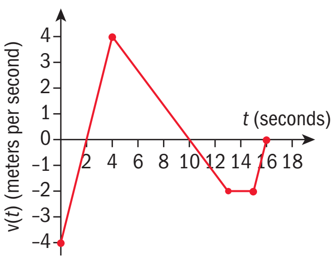
$12 = \frac{1}{3}(0)^3 - 9(0) + C \Rightarrow C = 12$

$s(t) = \frac{1}{3}t^3 - 9t + 12$

c distance travelled = $\int_2^8 \left|t^2 - 9\right| dt \approx 119$ m

6 a Acceleration at $t = 3$ is the gradient of $v(t)$ at $t = 3$. $a(3) = \frac{4-(-4)}{4-0} = 2 \text{ ms}^{-2}$

b The particle is traveling to the right when $v(t) > 0$ which is the interval $2 < t < 10$



c total distance traveled = $\int_0^{16} |v(t)| dt = \frac{1}{2}(2)(4)$

$+ \frac{1}{2}(8)(4)$

$+ \frac{1}{2}(6+2)(2)$

$= 28$ m

Exercise 9P

1 $\int_0^{10} 18.4e^{\frac{t}{20}} \mathrm{d}t \approx 239$ billions of barrels

2 $\int_0^{1.5} \left(1375t^2 - t^3\right) \mathrm{d}t \approx 1546$ spectators

3 $36.5 + \int_0^8 5te^{(-0.01t^4 + 0.13t^3 - 0.38t^2 - 0.3t + 0.9)} \mathrm{d}t \approx 240 \text{ cm}^3$

4 $4000 + \int_0^{20} -133\left(1 - \frac{t}{60}\right) \mathrm{d}t \approx 1780$ gallons

Review exercise

1 a $\int (4x^3 - 8x + 6)\mathrm{d}x = 4\left(\frac{1}{4}x^4\right) - 8\left(\frac{1}{2}x^2\right) + 6x + C$

$= x^4 - 4x^2 + 6x + C$

b $\int \sqrt[3]{x^4}\, \mathrm{d}x = \int x^{\frac{4}{3}} \mathrm{d}x = \frac{1}{\frac{7}{3}} x^{\frac{7}{3}} + C = \frac{3}{7}x^{\frac{7}{3}} + C$

c $\int \frac{3}{x^4} \mathrm{d}x = 3\int x^{-4} \mathrm{d}x = 3\left(-\frac{1}{3}x^{-3}\right) + C = -\frac{1}{x^3} + C$

d $\int \frac{5x^4 - 3x}{6x^2} \mathrm{d}x = \int \left(\frac{5}{6}x^2 - \frac{1}{2}\left(\frac{1}{x}\right)\right) \mathrm{d}x$

$= \frac{5}{6}\left(\frac{1}{3}x^3\right) - \frac{1}{2}\ln x + C$

$= \frac{5}{18}x^3 - \frac{1}{2}\ln x + C, x > 0$

e $\int e^{4x} \mathrm{d}x = \frac{1}{4}e^{4x} + C$

f $\int x^2(x^3 + 1)^4 \mathrm{d}x \Rightarrow u = x^3 + 1;$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2; \frac{1}{3}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = x^2$

$\int \left[\frac{1}{3}\left(\frac{du}{dx}\right)u^4\right] \mathrm{d}x = \frac{1}{3}\int u^4 \mathrm{d}u = \frac{1}{3}\left(\frac{1}{5}u^5\right) + C$

$= \frac{1}{15}u^5 + C = \frac{1}{15}(x^3 + 1)^5 + C$

g $\int \frac{1}{2x+3} \mathrm{d}x = \frac{1}{2}\ln(2x+3) + C, \ x > -\frac{3}{2}$

h $\int \frac{\ln x}{x} \mathrm{d}x \Rightarrow u = \ln x; \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \left(\ln x\left(\frac{1}{x}\right)\right) \mathrm{d}x = \int \left(u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right) \mathrm{d}x$

$= \int u \mathrm{d}u = \frac{1}{2}u^2 + C$

$= \frac{1}{2}(\ln x)^2 + C, \ x > 0$

i $\int (3x^2+1)(6x)\mathrm{d}x \Rightarrow u=3x^2+1;\ \frac{du}{dx}=6x$

$\int u\left(\frac{du}{dx}\right)\mathrm{d}x=\int u\ \mathrm{d}u=\frac{1}{2}u^2+C=\frac{1}{2}(3x^2+1)^2+C$

j $\int \frac{2e^x}{e^x+3}\mathrm{d}x \Rightarrow u=e^x+3;\ \frac{du}{dx}=e^x$

$\int \frac{2e^x}{e^x+3}\mathrm{d}x=2\int\left(\frac{1}{e^x+3}\right)(e^x)\mathrm{d}x$

$=2\int \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x=2\int\frac{1}{u}\mathrm{d}u$

$=2\ln u+C=2\ln(e^x+3)+C$

k $\int 3\sqrt{2x-5}\,\mathrm{d}x=3\int(2x-5)^{\frac{1}{2}}\mathrm{d}x$

$=3\left[\frac{1}{2}\left(\frac{2}{3}(2x-5)^{\frac{3}{2}}\right)\right]+C$

$=(2x-5)^{\frac{3}{2}}+C$

l $\int 2xe^{2x^2}\mathrm{d}x=2\int xe^{2x^2}\mathrm{d}x\Rightarrow$

$u=2x^2;\ \frac{du}{dx}=4x;\ \frac{1}{4}\left(\frac{du}{dx}\right)=x$

$2\int xe^{2x^2}\mathrm{d}x=2\int\frac{1}{4}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)e^u\mathrm{d}x$

$=\frac{1}{2}\int e^u\mathrm{d}u=\frac{1}{2}e^u+C=\frac{1}{2}e^{2x^2}+C$

2 a $\int_0^2(3x^2-6)\mathrm{d}x=\left[3\left(\frac{1}{3}x^3\right)-6x\right]_0^2=\left[x^3-6x\right]_0^2$

$=\left[(2)^3-6(2)\right]-\left[(0)^3-6(0)\right]=-4$

b $\int_4^{16}\frac{4}{\sqrt{t}}\mathrm{d}t=4\int_4^{16}t^{-\frac{1}{2}}\mathrm{d}t=4\left[2t^{\frac{1}{2}}\right]_4^{16}$

$=\left[8t^{\frac{1}{2}}\right]_4^{16}=\left[8(16)^{\frac{1}{2}}\right]-\left[8(4)^{\frac{1}{2}}\right]$

$=32-16=16$

c $\int_1^{e^2}\frac{4}{x}\mathrm{d}x=4\int_1^{e^2}\frac{1}{x}\mathrm{d}x=4\left[\ln x\right]_1^{e^2}$

$=4\left[\ln e^2\right]-4\left[\ln 1\right]$

$=4(2)-4(0)=8$

d $\int_0^1 6xe^{3x^2+3}\mathrm{d}x\Rightarrow u=3x^2+3;\ \frac{du}{dx}=6x;$

when $x=0$ then $u=3(0)^2+3=3$ and
when $x=1$ then $u=3(1)^2+3=6$

$\int_0^1 6xe^{3x^2+3}\mathrm{d}x=\int_{x=0}^{x=1}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)e^u\mathrm{d}x$

$=\int_{u=3}^{u=6}e^u\mathrm{d}u=\left[e^u\right]_3^6=e^6-e^3$

e $\int_{-1}^1(3x-1)^3\mathrm{d}x=\frac{1}{3}\left[\frac{1}{4}(3x-1)^4\right]_{-1}^1=\left[\frac{1}{12}(3x-1)^4\right]_{-1}^1$

$=\left[\frac{1}{12}(3(1)-1)^4\right]-\left[\frac{1}{12}(3(-1)-1)^4\right]$

$=\frac{16}{12}-\frac{256}{12}=-20$

f $\int_0^2\frac{1}{2x+1}\mathrm{d}x=\frac{1}{2}\left[\ln(2x+1)\right]_0^2$

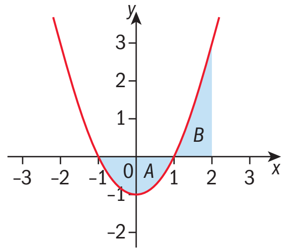
$=\frac{1}{2}\left[\ln(2(2)+1)\right]-\frac{1}{2}\left[\ln(2(0)+1)\right]$

$=\frac{1}{2}(\ln 5)-\frac{1}{2}\ln 1=\frac{\ln 5}{2}$

3 $f(x)=x^2-1$

a Area of region B

$=\int_1^2(x^2-1)\mathrm{d}x$



b Area of region B

$=\int_1^2(x^2-1)\mathrm{d}x=\left[\frac{1}{3}x^3-x\right]_1^2$

$=\left[\frac{1}{3}(2)^3-2\right]-\left[\frac{1}{3}(1)^3-1\right]$

$=\frac{2}{3}-\left(-\frac{2}{3}\right)=\frac{4}{3}$

c $\int_1^2(x^2-1)\mathrm{d}x-\int_{-1}^1(x^2-1)\mathrm{d}x$

d $\pi\int_1^2(x^2-1)^2\mathrm{d}x$

4 $f'(x)=3x-2;\ (2,6)$

$f(x)=\int(3x-2)\mathrm{d}x=3\left(\frac{1}{2}x^2\right)-2x+C$

$=\frac{3}{2}x^2-2x+C$

$\frac{3}{2}(2)^2-2(2)+C=6\Rightarrow 6-4+C=6\Rightarrow C=4$

$f(x)=\frac{3}{2}x^2-2x+4$

5 $\int_1^5 f(x)\mathrm{d}x=20$

a Given $\int_1^5 f(x)\mathrm{d}x=20$.

$\int_1^5\frac{1}{4}f(x)\mathrm{d}x=\frac{1}{4}\int_1^5 f(x)\mathrm{d}x=\frac{1}{4}(20)=5;$

b $\int_1^5 [f(x)+2]\,dx = \int_1^5 f(x)\,dx + \int_1^5 2\,dx$

$= 20 + [2x]_1^5 = 20 + (10-2) = 28$

6 $v(t) = 4e^{2t} + 2;\ s(0) = 8$

$s(t) = \int (4e^{2t}+2)\,dt = 4\left(\frac{1}{2}e^{2t}\right) + 2t + C$

$= 2e^{2t} + 2t + C$

$2e^{2(0)} + 2(0) + C = 8 \Rightarrow 2 + C = 8 \Rightarrow C = 6$

$s(t) = 2e^{2t} + 2t + 6$

7 $\int_1^k \frac{1}{2x-1}\,dx = \ln 5$

$\int_1^k \frac{1}{2x-1}\,dx = \left[\frac{1}{2}\ln(2x-1)\right]_1^k$

$= \left[\frac{1}{2}\ln(2(k)-1)\right] - \left[\frac{1}{2}\ln(2(1)-1)\right]$

$= \frac{1}{2}\ln(2k-1)$

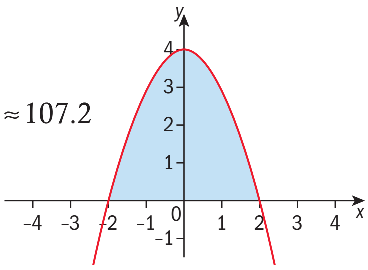
$\frac{1}{2}\ln(2k-1) = \ln 5 \Rightarrow \ln\sqrt{2k-1} = \ln 5 \Rightarrow$

$\sqrt{2k-1} = 5 \Rightarrow 2k-1 = 25 \Rightarrow k = 13$

Review exercise

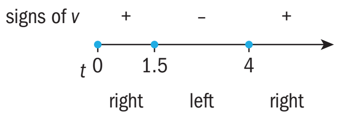
1 $f(x) = 4 - x^2$

$V = \int_{-2}^{2} \pi\left(4-x^2\right)^2 dx \approx 107.2$



2 a $v(t) = 2t^2 - 11t + 12,\ t \geq 0$

$a(t) = v'(t) = 4t - 11$



b Particle moves left for $a < t < b$

$v(t) = 2t^2 - 11t + 12$

$2t^2 - 11t + 12 = 0 \Rightarrow x = 1.5, 4$

$a = 1.5$ and $b = 4$

c $\int_2^5 \left|2t^2 - 11t + 12\right| dt \approx 7.83$ m

3 a $f(x) = x^3 - 2 \Rightarrow f(-1) = -3$

$f'(x) = 3x^2 \Rightarrow m = f'(-1) = 3$

$y + 3 = 3(x+1)$ or $y = 3x$

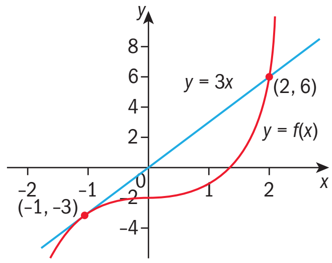
b Use a GDC to solve : $x^3 - 2 = 3x$

$x = 2$

$f(2) = 2^3 - 2 = 6$

The point is (2, 6)

c



d $\int_{-1}^{2} \left[3x - (x^3 - 2)\right] dx = 6.75$

10 Bivariate analysis

Answers

Skills check

Evaluate

1 $2^5 = 32$

2 $3^3 = 27$

3 $7^3 = 343$

4 $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$

5 $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$

6 $0.001^3 = 0.000\ 000\ 001$

State the value of n in the following equations

1 $2^n = 16$ $\quad n = 4$

2 $3^n = 243$ $\quad n = 5$

3 $7^n = 343$ $\quad n = 3$

4 $5^n = 625$ $\quad n = 4$

5 $(-4)^n = -64$ $\quad n = 3$

6 $\left(\frac{1}{2}\right)^n = \frac{1}{8}$ $\quad n = 3$

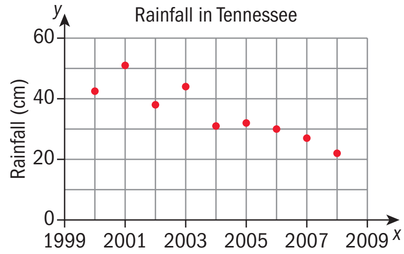
Exercise 10A

1 **a** Positive Strong
  b Negative Weak
  c Negative Strong
  d Positive Weak
  e No correlation

2 **a** **i** positive, **ii** linear, **iii** strong,
  b **i** negative, **ii** linear, **iii** strong,
  c **i** positive, **ii** linear, **iii** Moderate.
  d **i** No association, **ii** Non linear, **iii** zero.
  e **i** positive, **ii** linear, **iii** weak.
  f **i** Negative, **ii** non linear, **iii** strong.

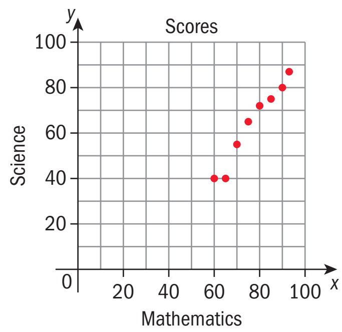
3 **a** If the independent and dependent variables show a positive correlation then as the independent variable increases the dependent variable increases.
  b If the independent and dependent variables show a negative correlation then as the independent variable increases the dependent variable decreases.

4 **a**



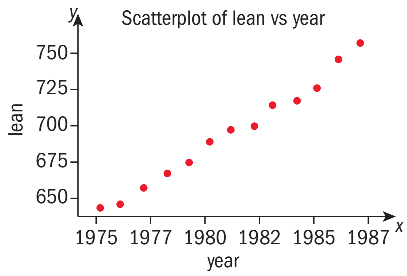
  b Strong, negative.
  c As the years increase the rainfall decreases

5 **a**



  b Strong, positive, linear.

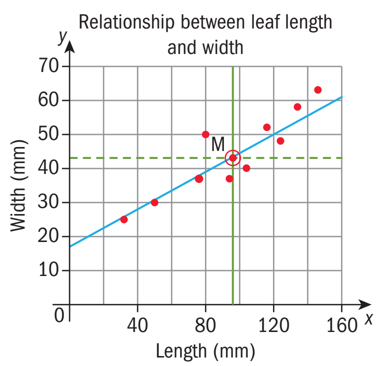
6



  a Strong, positive.
  b The lean is increasing as the years increase.

Exercise 10B

1 **a** Mean point = (mean of x, mean of y)
  = (96.7, 44.1)
  b

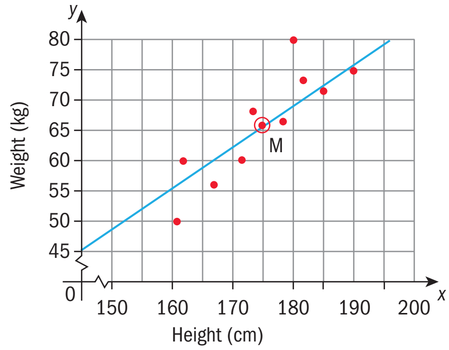


2 **a** **i** $\dfrac{182+173+162+178+190+161+180+172+167+185}{10}$

$= 175$ cm

ii $\dfrac{73+68+60+66+75+50+80+60+56+72}{10} = 66$ kg

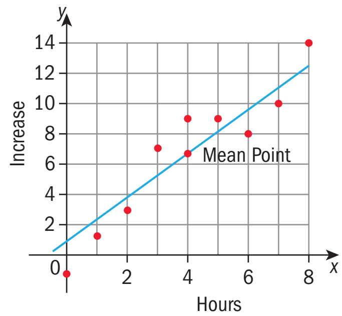
b



3 **a** Mean point = (mean of x, mean of y)

$= (4, 6.67)$

b



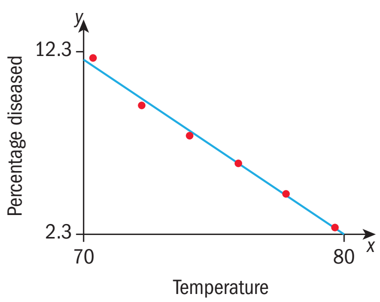
c Strong, positive

d An increase in the number of hours spent studying mathematics produces an increase in the grade.

Exercise 10C

1

Temperature, (x) °F	70	72	74	76	78	80
Percentage of diseased leaves, (y)	12.3	9.5	7.7	6.1	4.3	2.3



a $(\bar{x}, \bar{y}) = (75, 7.03)$

b $y = -0.96x + 79$

c % diseased = (–0.96 × Temperature) + 79

% diseased = (–0.96 × 79) + 79 = 3.2%

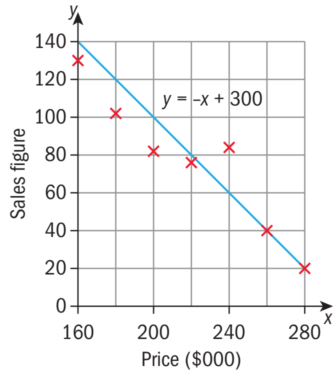
2 **a** Mean house price = $\dfrac{\text{sum of prices}}{\text{number of prices}} = \dfrac{1540}{7} = 220$

The mean house price is $220 000

b Mean sales = $\dfrac{\text{sum of sales}}{\text{number of sales}} = \dfrac{528}{7} = 75.4$

The mean number of sales is estimated at 75.4

c and **d** Note the values of m and b in the equation $y = mx + b$ are approximate.



e Approximately 70 houses.

Exercise 10D

1 The slope is –0.3. As a student plays one more day of sport per year they do 18 mins less homework per week.

The y-intercept is 40, which means that the average student who does no sport does 40 hours of homework per week.

2 The slope is 6. For every criminal a person knows, they will generally have been convicted of 6 more crimes.

The y-intercept is 0.5, which means that people who do not know any criminals will, on average, have been convicted 0.5 times.

3 The slope is 2.4. For every pack of cigarettes smoked per week a person will, on average, take 2.4 more sick days per year.

The y-intercept is 7, which means that the average person that does not smoke has 7 sick days per year.

4 The slope is 100. 100 more customers come to his shop every year.

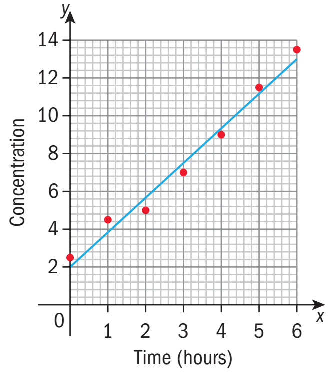
The y-intercept is –5, which means that –5 people visited his shop in year zero, the y-intercept is not suitable for interpretation.

5 The slope is 0.8. Every 1 mark increase in mathematics results in a 0.8 increase in science.

The y-intercept is –10 which is not suitable for interpretation as a zero in mathematics would mean a –10 in science.

Exercise 10E

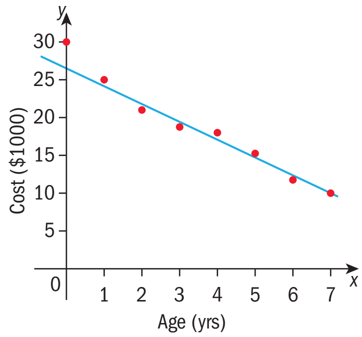
1 a



b $y = 1.84x + 1.99$

c Concentration after 3.5 hours:
$y = 1.99 + 1.84 \times 3.5 = 8.43$

2 a

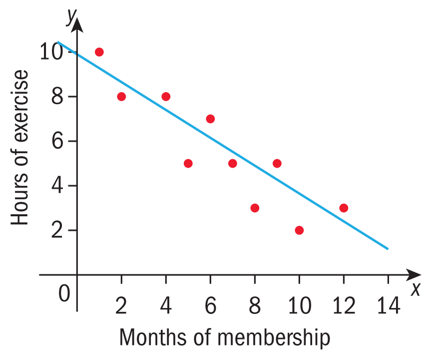


b $y = -2.67x + 28.1$

c Cost = (–2.67 × Age) + 28.1
= (–2.67 × 4.5) + 28.1 = MYR16 085

d The relationship may not be linear. Old cars are often more expensive after 50 yrs than when new.

3 a



b $y = -0.665x + 9.86$

c Hours of exercise = (–0.665 × months of membership) + 9.86 = (–0.665 × 3) + 9.86 = 7.865 hrs.

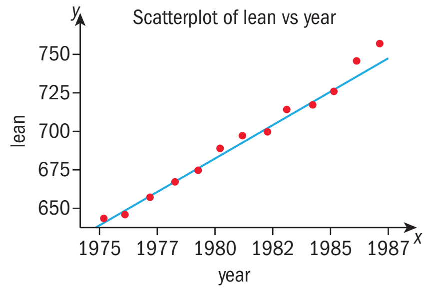
d No. The equation gives –6.1 hrs of exercise!

4 Fifty years = 600 months, and the line would predict Sarah's height at 50 years to be about 302 cm = 3.02 meters. Clearly there is a major difficulty with extrapolation. In fact, most females reach their maximum height in their mid to late teens, and from then on, their height is fairly constant and therefore extrapolating with a linear function is unsuitable.

5 Revisit the data from the leaning tower of Pisa.

a (1981, 694)

b



d $y = 9.32x - 17\,767$

e Lean = (9.32 × year) – 17 767
= (9.32 × 1990) – 17 767 = 780

Exercise 10F

1 $r = 0.863$. There is a strong, positive correlation.

2 a 0.789

b Strong, positive correlation.

c The income increases as the number of years of education increases.

3 a 0.907

b The stopping distance increases, as the car gets older.

c Strong positive correlation.

4 a –0. 887

b Strong, negative correlation.

c Yes, Kelly's grade would increase if the chat time decreased.

5 a 0.026

b Positive, weak correlation.

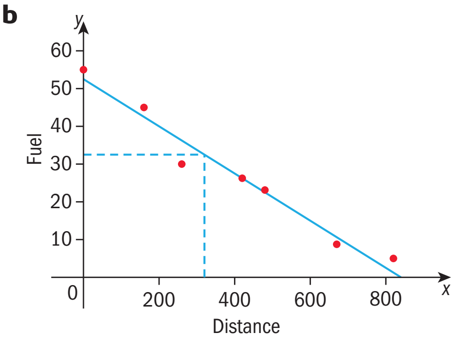
c No. Mo's grade would not increase if the game time decreased.

6 $r = 0.994$. Strong, positive correlation.

Review exercise

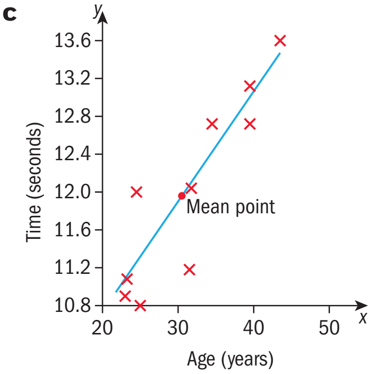
1 a ii **b** v **c** iii **d** i

2 a b



c 32. See the dotted lines on the graph.

3 **a, c**



b Mean Age $= \dfrac{\text{sum of ages}}{\text{number of policemen}} = \dfrac{340}{10} = 34$

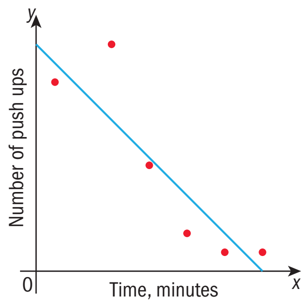
Mean time $= \dfrac{\text{sum of times}}{\text{number of policemen}} = \dfrac{120}{10} = 12$

Mean age = 34 years. Mean time = 12 secs.

d Approximately 11.7 secs.

Review exercise

1 **a**



b As the time increases, the number of push-ups decreases.

c $y = -1.29x + 9$

d $r = -0.929$. There is a strong, negative correlation.

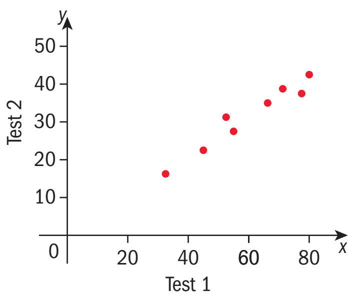
2 **a** $w = -22.4 + 55.5h$

b $w = -22.4 + 55.5 \times 1.6 = 66.4$ kg

3 **a** $r = 0.785$ **b** $y = 30.7 + 0.688x$

c IQ = 30.6 + (0.688 × 100) = 99.4. This should be reasonably accurate since the product moment correlation coefficient shows fairly strong correlation.

4 **a**



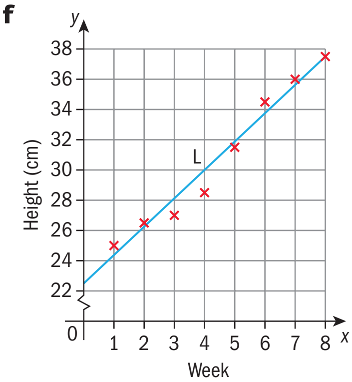
b Positive, strong.

c Students with a high score on test 1 tend to have a high score on test 2.

d $y = 0{\cdot}50x + 0{\cdot}48$

e Test 2 Score = (0.50 × Test 1 score) + 0.48
= (0.50 × 40) + 0.48 = 20.48

5 **a, c, f**



b (4, 30)

d **i** $r = 0.986$

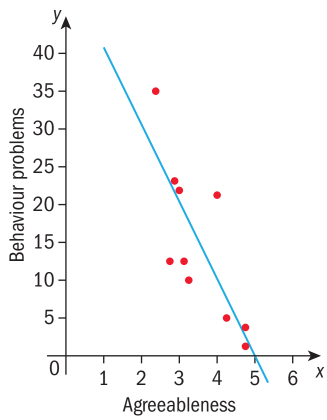
ii (very) strong positive correlation

e $y = 1.83x + 22.7$

g Height = (1.83 × week number) + 22.7
= (1.83 × 4.5) + 22.7 = 30.9 cm

h Not possible to find an answer as the value lies too far outside the given set of data.

6 **a**



b Behavior problems decrease

c –0.797

d Strong, negative correlation.

e Teenagers who were more agreeable tended to have fewer behavior problems.

f $y = -10.2x + 51.0$

g Number of behavior problems
= (–10.2 × Agreeableness score) + 51.0
= (–10.2 × 4.5)+51.0 = 5.1

7 **a** $y = 10.7x + 121$ (3sf)

b **i** Every coat on average costs $10.66 to produce,

ii When the factory does not produce any clothes then $x = 0$, it has to pay costs of $121.

c Cost = (10.7 × 70) + 121 = $870

d $19.99x > 10.66x + 121$

$9.33x > 121$

$x > 12.969$

13 coats should be produced in one day in order to make a profit.

11 Trigonometry

Skills check

1 **a** $x° + 49° + 41° = 180°$

$x = 90°$

b $x° + 2x° + (x - 20)° = 4x° - 20° = 180°$

$x = 50°$

c $x° + 56° + 56° = 180°$

$x = 68°$

d $(4x)° + (x + 20)° + (x + 20)° = 6x° + 40° = 180°$

$x = \frac{70}{3}°$

e $x^2 = (2.4)^2 + (5.6)^2 = 37.12$

$x = \sqrt{37.12} \approx 6.09$

f $x^2 + (19)^2 = (24)^2$

$x^2 = (24)^2 - (19)^2 = 215$

$x = \sqrt{215} \approx 14.7$

Exercise 11A

1 $12^2 + b^2 = 20^2 \to b^2 = 20^2 - 12^2 = 256$

$b = \sqrt{256} = 16$ cm

$\sin A = \frac{12}{20} \to A = \sin^{-1}\left(\frac{12}{20}\right) \approx 36.9°$

$\cos B = \frac{12}{20} \to B = \cos^{-1}\left(\frac{12}{20}\right) \approx 53.1°$

2 $B = 180 - 40 - 90 = 50°$

$\tan 40 = \frac{a}{37} \to a = 37 \tan 40 \approx 31.0$ cm

$\cos 40 = \frac{37}{c} \to c = \frac{37}{\cos 40} \approx 48.3$ cm

3 $A = 180 - 55 - 90 = 35°$

$\cos 55 = \frac{a}{4.5} \to a = 4.5 \cos 55 \approx 2.58$ cm

$\sin 55 = \frac{b}{4.5} \to b = 4.5 \sin 55 \approx 3.69$ cm

4 $a^2 + 48^2 = 60^2 \to a^2 = 60^2 - 48^2 = 1296$

$a = \sqrt{1296} = 36$

$\cos A = \frac{48}{60} \to A = \cos^{-1}\left(\frac{48}{60}\right) \approx 36.9°$

$\sin B = \frac{48}{60} \to B = \sin^{-1}\left(\frac{48}{60}\right) \approx 53.1°$

5 $B = 180 - 35 - 90 = 55°$

$\tan 35 = \frac{11}{b} \to b = \frac{11}{\tan 35} \approx 15.7$ cm

$\sin 35 = \frac{11}{c} \to c = \frac{11}{\sin 35} \approx 19.2$ cm

6 $c^2 = (8.5)^2 + (9.7)^2 = 166.34$

$c = \sqrt{166.34} \approx 12.9$ cm

$\tan A = \frac{8.5}{9.7} \to A = \tan^{-1}\left(\frac{8.5}{9.7}\right) \approx 41.2°$

$\tan B = \frac{9.7}{8.5} \to B = \tan^{-1}\left(\frac{9.7}{8.5}\right) \approx 48.8°$

7 $(2x)^2 + (5x - 1)^2 = (x^2 + 1)^2$

$4x^2 + 25x^2 - 10x + 1 = x^4 + 2x^2 + 1$

$x^4 - 27x^2 + 10x = 0$

using GDC, $x = 5$

$\tan A = \frac{10}{24} \to A = \tan^{-1}\left(\frac{10}{24}\right) \approx 22.6°$

$\tan B = \frac{24}{10} \to B = \tan^{-1}\left(\frac{24}{10}\right) \approx 67.4°$

Exercise 11B

Note: many of the questions in this section can be answered in your head, if you remember the patterns of these special right triangles.

1 **a** $12^2 + b^2 = 24^2 \to b^2 = 24^2 - 12^2 = 432$

$b = 432 = 12\sqrt{3}$ cm

$\sin A = \frac{12}{24} = \frac{1}{2} \to A = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

$\cos B = \frac{12}{24} = \frac{1}{2} \to B = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

b $B = 180 - 45 - 90 = 45°$

$\tan 45 = \frac{a}{9} \to a = 9 \tan 45 = 9$ cm

$\sin 45 = \frac{9}{c} \to c = \frac{9}{\sin 45} = \frac{9}{\left(\frac{1}{\sqrt{2}}\right)} = 9\sqrt{2}$ cm

c $A = 180 - 60 - 90 = 30°$

$\sin 30 = \frac{a}{4.5} \to a = 4.5 \sin 30 = 2.25$ cm

$\sin 60 = \frac{b}{4.5} \to b = 4.5 \sin 60 = \frac{9}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{4}$ cm

d $a^2 + 6^2 = \left(4\sqrt{3}\right)^2 \to a^2 = \left(4\sqrt{3}\right)^2 - 6^2 = 12$

$a = \sqrt{12} = 2\sqrt{3}$ cm

$\cos A = \frac{6}{4\sqrt{3}} \to A = \cos^{-1}\left(\frac{6}{4\sqrt{3}}\right) = 30°$

$B = 180 - 90 - 30 = 60°$

e $\left(5\sqrt{2}\right)^2 + b^2 = (10)^2$

$\to b^2 = (10)^2 - \left(5\sqrt{2}\right)^2 = 100 - 50 = 50$

$b = \sqrt{50} = 5\sqrt{2}$ cm

$\tan A = \frac{5\sqrt{2}}{5\sqrt{2}} = 1 \to A = \tan^{-1}(1) = 45°$

$B = 180 - 90 - 45 = 45°$

2 $x^2 = (8)^2 + (8)^2 = 128$

$x = \sqrt{128} = 8\sqrt{2}$ cm

$\tan 30 = \frac{1}{\sqrt{3}} = \frac{8}{y+8} \to y + 8 = 8\sqrt{3}$

$y = 8\sqrt{3} - 8$ cm

$\sin 30 = \frac{1}{2} = \frac{8}{z} \to z = 16$ cm

3 **a** $\sin 60 = \frac{\sqrt{3}}{2} = \frac{x+2}{x^2-4} = \frac{1}{x-2} \to \sqrt{3}(x-2) = 2$

$x\sqrt{3} - 2\sqrt{3} = 2 \to x\sqrt{3} = 2 + 2\sqrt{3} \to x = \frac{2+2\sqrt{3}}{\sqrt{3}}$

b $\tan 60 = \sqrt{3} = \frac{x+2}{AC}$

$\to \sqrt{3}AC = x + 2 = \frac{2+2\sqrt{3}}{\sqrt{3}} + 2 = \frac{2+2\sqrt{3}}{\sqrt{3}} + \frac{2\sqrt{3}}{\sqrt{3}}$

$= \frac{2+4\sqrt{3}}{\sqrt{3}}$

$AC = \frac{2+4\sqrt{3}}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) = \frac{2+4\sqrt{3}}{3}$ cm

4 $\tan 45 = 1 = \frac{4x-1}{x^2+2} \to x^2 + 2 = 4x - 1$

$\to x^2 - 4x + 3 = 0$

$x = 1, 3$

$\sin 45 = \frac{1}{\sqrt{2}} = \frac{4x-1}{AB} \to AB = \sqrt{2}(4x-1)$

If $x = 1$, $AB = \sqrt{2}(4(1)-1) = \sqrt{2}(3) = 3\sqrt{2}$ cm

If $x = 3$, $AB = \sqrt{2}(4(3)-1) = \sqrt{2}(11) = 11\sqrt{2}$ cm

5 $w^2 = 4^2 + 9^2 = 16 + 81 = 97 \to w = \sqrt{97}$

$x = \frac{w}{\sin 45} = w\sqrt{2} = \sqrt{97}\sqrt{2} = \sqrt{194}$

$\tan 65 = \frac{x}{y} = \frac{\sqrt{194}}{y} \to y = \frac{\sqrt{194}}{\tan 65}$

$\sin 65 = \frac{x}{z} = \frac{\sqrt{194}}{z} \to z = \frac{\sqrt{194}}{\sin 65}$

$w \approx 9.8$ cm, $x \approx 13.9$ cm, $y \approx 6.5$ cm, $z \approx 15.4$ cm

Exercise 11C

1 **a** $h = \sqrt{15^2 - 5^2} = \sqrt{200} = 10\sqrt{2}$ cm

b $\cos B\hat{A}C = \frac{5}{15} \to B\hat{A}C = \cos^{-1}\frac{5}{15} \approx 70.5°$

$A\hat{B}C = 180 - B\hat{A}C - B\hat{C}A$

$= 180 - 2B\hat{A}C \approx 38.9°$

2 **a** $AE = \sqrt{28^2 + 8^2} = \sqrt{848} \approx 29.1$ cm

$BE = \sqrt{28^2 + 20^2} = \sqrt{1184} \approx 34.4$ cm

b $\tan A\hat{E}D = \frac{28}{8} \to A\hat{E}D = \tan^{-1}\left(\frac{28}{8}\right) \approx 74.1°$

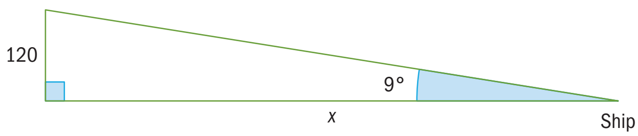
$\tan E\hat{B}A = \frac{28}{20} \to E\hat{B}A = \tan^{-1}\left(\frac{28}{20}\right) \approx 54.5°$

$E\hat{A}B = A\hat{E}D$ (alternate angles), so

$A\hat{E}B = 180 - E\hat{A}B - E\hat{B}A$

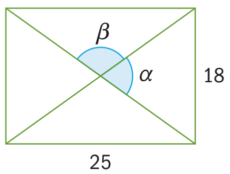
$= 180 - A\hat{E}D - E\hat{B}A \approx 51.5°$

3



$\tan 9 = \frac{120}{x} \to x = \frac{120}{\tan 9} \approx 758$ m

4



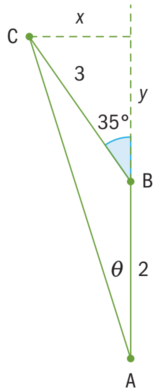
$\tan\left(\frac{\alpha}{2}\right) = \frac{9}{12.5} \to \frac{\alpha}{2} = \tan^{-1}\left(\frac{9}{12.5}\right)$

$\to \alpha = 2\tan^{-1}\left(\frac{9}{12.5}\right) \approx 71.5°$

$\tan\left(\frac{\beta}{2}\right) = \frac{12.5}{9} \to \frac{\beta}{2} = \tan^{-1}\left(\frac{12.5}{9}\right)$

$\to \beta = 2\tan^{-1}\left(\frac{12.5}{9}\right) \approx 108.5°$

5



$\sin 35 = \frac{x}{3} \to x = 3\sin 35 \approx 1.7207$

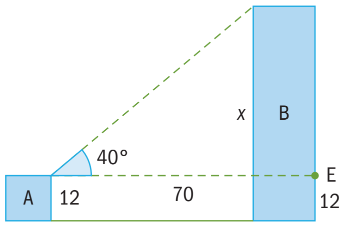
$\cos 35 = \frac{y}{3} \to y = 3\cos 35 \approx 2.457456$

$AC^2 = x^2 + (y+2)^2 \to AC \approx 4.778$

$\sin\theta = \frac{x}{AC} \approx \frac{1.7207}{4.778} \to \theta \approx 21.1°$

4.78 km, N21.1°W

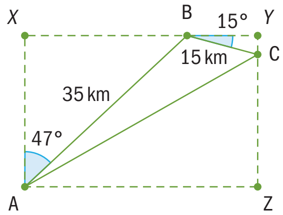
6



$\tan 40 = \frac{x}{70} \to x = 70\tan 40 \approx 58.737$

height $= x + 12 \approx 70.7$ m

7



$\sin 47 = \frac{BX}{AB} = \frac{BX}{35} \to BX = 35\sin 47 \approx 25.597$

$\cos 47 = \frac{AX}{AB} = \frac{AX}{35} \to AX = 35\cos 47 \approx 23.8699$

$\sin 15 = \frac{YC}{BC} = \frac{YC}{15} \to YC = 15\sin 15 \approx 3.882$

$\cos 15 = \frac{BY}{BC} = \frac{BY}{15} \to BY = 15\cos 15 \approx 14.4889$

$AZ = BX + BY \approx 40.086$

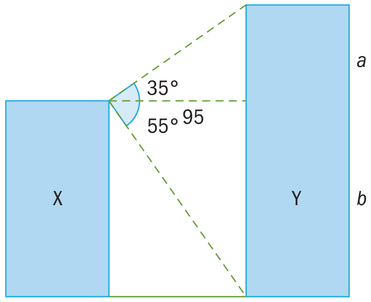
$CZ = AX - YC \approx 19.988$

$AC = \sqrt{AZ^2 + CZ^2} \approx 44.793$

$A\hat{C}Z = \tan^{-1}\left(\frac{AZ}{CZ}\right) \approx 63.498°$

44.8 km, bearing approx. $(180 + 63.5)° = 243.5°$

8

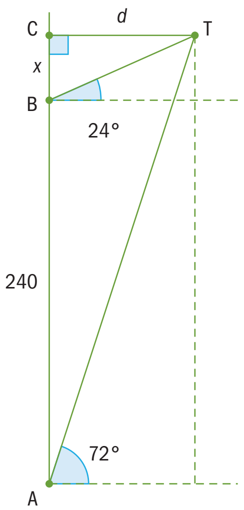


$\tan 35 = \frac{a}{95} \rightarrow a = 95\tan 35 \approx 66.5197$

$\tan 55 = \frac{b}{95} \rightarrow b = 95\tan 55 \approx 135.674$

X is 135.7 m tall, Y is (135.7 + 66.5) = 202.2 m tall

9



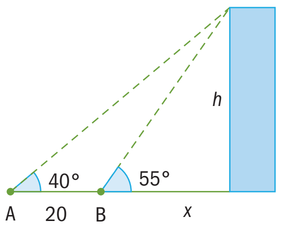
$\tan 72 = \frac{x+240}{d} \rightarrow x = d\tan 72 - 240$

$\tan 24 = \frac{x}{d} \rightarrow x = d\tan 24$

$d\tan 24 = d\tan 72 - 240 \rightarrow d(\tan 72 - \tan 24) = 240$

$d = \frac{240}{\tan 72 - \tan 24} \approx 91.2$ m

10



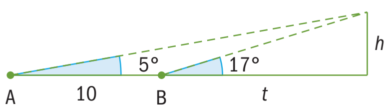
$\tan 40 = \frac{h}{x+20} \rightarrow x = \frac{h}{\tan 40} - 20$

$\tan 55 = \frac{h}{x} \rightarrow x = \frac{h}{\tan 55}$

$\frac{h}{\tan 55} = \frac{h}{\tan 40} - 20 \rightarrow h\left(\frac{1}{\tan 40} - \frac{1}{\tan 55}\right) = 20$

$h = \frac{20}{\left(\frac{1}{\tan 40} - \frac{1}{\tan 55}\right)} \approx 40.7$ cm

11



$\tan 5 = \frac{h}{t+10} \rightarrow h = (t+10)\tan 5$

$\tan 17 = \frac{h}{t} \rightarrow h = t\tan 17$

$\rightarrow (t+10)\tan 5 = t\tan 17$

$t(\tan 17 - \tan 5) = 10\tan 5$

$t = \frac{10\tan 5}{\tan 17 - \tan 5} \approx 4.01$ s

10 and *t* in the diagram represent 10*v* and *tv* (the distance travelled in 10 seconds and in *t* seconds).

12 **a** $\tan H\hat{A}D = \frac{9}{24} \rightarrow H\hat{A}D = \tan^{-1}\frac{9}{24} \approx 20.6°$

b $\tan A\hat{B}E = \frac{9}{18} \rightarrow A\hat{B}E = \tan^{-1}\frac{9}{18} \approx 26.6°$

c $HA = \sqrt{9^2 + 24^2} = \sqrt{657}$

$\tan G\hat{A}H = \frac{18}{\sqrt{657}}$

$\rightarrow G\hat{A}H = \tan^{-1}\left(\frac{18}{\sqrt{657}}\right) \approx 35.1°$

d $DG = \sqrt{9^2 + 18^2} = \sqrt{405}$

$\tan A\hat{G}D = \frac{24}{\sqrt{405}}$

$\rightarrow A\hat{G}D = \tan^{-1}\left(\frac{24}{\sqrt{405}}\right) \approx 50.0°$

Exercise 11D

1 **a** $(\cos 20, \sin 20) \rightarrow (0.940, 0.342)$

b $(\cos 17, \sin 17) \rightarrow (0.956, 0.292)$

c $(\cos 60, \sin 60) \rightarrow (0.5, 0.866)$

d $(\cos 74, \sin 74) \rightarrow (0.276, 0.961)$

e $(\cos 90, \sin 90) \rightarrow (0, 1)$

2 **a** $\cos^{-1} 0.408 \approx 66°$ or $\sin^{-1} 0.913 \approx 66°$

b $\cos^{-1} 0.155 \approx 81°$ or $\sin^{-1} 0.988 \approx 81°$

c $\cos^{-1} 0.707 \approx 45°$ or $\sin^{-1} 0.707 \approx 45°$

d $\cos^{-1} 0.970 \approx 14°$ or $\sin^{-1} 0.242 \approx 14°$

3 **a** $A = \frac{1}{2}(\cos 70)(\sin 70) \approx 0.161$

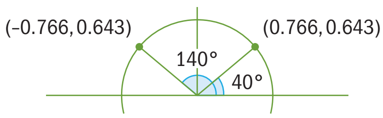
b $A = \frac{1}{2}(\cos 38)(\sin 70) \approx 0.243$

c $A = \frac{1}{2}(\cos 24)(\sin 70) \approx 0.186$

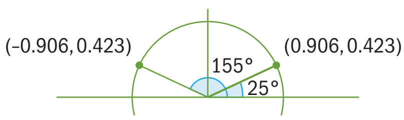
d $A = \frac{1}{2}(\cos 30)(\sin 70) \approx 0.217$

Investigation – obtuse angles

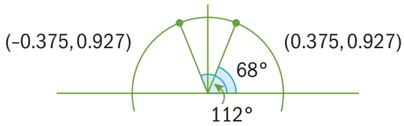
1



2



3



Exercise 11E

1 **a** $B(\cos 30, \sin 30) \rightarrow B(0.866, 0.5)$, $C(-0.866, 0.5)$

b $B(\cos 57, \sin 57) \to B(0.545, 0.839)$, $C(-0.545, 0.839)$

c $B(\cos 45, \sin 45) \to B(0.707, 0.707)$, $C(-0.707, 0.707)$

d $B(\cos 13, \sin 13) \to B(0.974, 0.225)$, $C(-0.974, 0.225)$

e $B(\cos 85, \sin 85) \to B(0.087, 0.996)$, $C(-0.087, 0.996)$

2 **a** $\cos^{-1}(-0.332) \approx 109.4° \to 180 - 109.4 = 70.6°$

b $\cos^{-1}(-0.955) \approx 162.7° \to 180 - 162.7 = 17.3°$

c $\cos^{-1}(-0.903) \approx 154.6° \to 180 - 154.6 = 25.4°$

d $\cos^{-1}(-0.769) \approx 140.3° \to 180 - 140.3 = 39.7°$

3 **a** $\sin 15 \approx 0.2588$, $180 - 15 = 165°$

b $\sin 36 \approx 0.5878$, $180 - 36 = 144°$

c $\sin 81 \approx 0.9877$, $180 - 81 = 99°$

d $\sin 64 \approx 0.8988$, $180 - 64 = 116°$

4 **a** $\sin^{-1} 0.871 \approx 60.6°$, $180 - 60.6 = 119.4°$

b $\sin^{-1} 0.436 \approx 25.8°$, $180 - 25.8 = 154.2°$

c $\sin^{-1} 0.504 \approx 30.3°$, $180 - 30.3 = 149.7°$

d $\sin^{-1} 0.5 \approx 30°$, $180 - 30 = 150°$

Exercise 11F

1 **a** $\tan 56.3 \approx 1.50$

b $\tan 117.5 \approx -1.92$

c $\tan 137.7 \approx -0.910$

d $\tan 45 = 1$

2 **a** $\tan\theta = \frac{0.738}{0.674} \to y = 1.09x,\ \theta \approx 48°$

b $\tan\theta = \frac{0.882}{0.471} \to = 1.87x,\ \theta \approx 62°$

c $\tan\theta = \frac{0.942}{-0.336} \to y = -2.80x,\ \theta \approx 110°$

d $\tan\theta = \frac{-1.64}{1.35} \to y = -1.21x,\ \theta \approx 129°$

e $y = -0.75x,\ \theta = 180° - \alpha,$

$\tan\alpha = \frac{0.6}{0.8} \Rightarrow \alpha = 36.9$

$\Rightarrow \theta = 180 - 36.9 = 143°$

f $\tan\theta = \frac{3.76}{1.59} \to y = 2.36x,\ \theta \approx 113°$

Exercise 11G

1 **a** $\hat{C} = 180 - 47 - 83 = 50°$

$\frac{\sin 47}{a} = \frac{\sin 83}{24} \to a = \frac{24\sin 47}{\sin 83} \approx 17.7$ cm

$\frac{\sin 50}{c} = \frac{\sin 83}{24} \to c = \frac{24\sin 50}{\sin 83} \approx 18.5$ cm

b $\hat{B} = 180 - 40 - 72 = 68°$

$\frac{\sin 40}{a} = \frac{\sin 72}{2.5} \to a = \frac{2.5\sin 40}{\sin 72} \approx 1.69$ cm

$\frac{\sin 68}{b} = \frac{\sin 72}{2.5} \to b = \frac{2.5\sin 68}{\sin 72} \approx 2.44$ cm

c $\frac{\sin 55}{4.5} = \frac{\sin\hat{B}}{3.6} \to \sin\hat{B} = \frac{3.6\sin 55}{4.5}$

$\to \hat{B} = \sin^{-1}\left(\frac{3.6\sin 55}{4.5}\right) \approx 40.9°$

$\hat{C} \approx 180 - 55 - 40.9 \approx 84.1°$

$\frac{\sin 84.056}{c} = \frac{\sin 55}{4.5} \to c = \frac{4.5\sin 84.056}{\sin 55} \approx 5.46$ cm

d $\hat{A} = 180 - 15 - 125 = 40°$

$\frac{\sin 40}{a} = \frac{\sin 15}{60} \to a = \frac{60\sin 40}{\sin 15} \approx 149$ cm

$\frac{\sin 125}{c} = \frac{\sin 15}{60} \to c = \frac{60\sin 125}{\sin 15} \approx 190$ cm

e $\hat{C} = 180 - 27 - 43 = 110°$

$\frac{\sin 27}{a} = \frac{\sin 110}{5.8} \to a = \frac{5.8\sin 27}{\sin 110} \approx 2.80$ cm

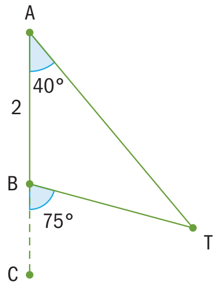
$\frac{\sin 43}{b} = \frac{\sin 110}{5.8} \to b = \frac{5.8\sin 43}{\sin 110} \approx 4.21$ cm

2 $Y\hat{X}Z = 180 - 68.2 - 68.2 = 43.6°$

$\frac{\sin 43.6}{20} = \frac{\sin 68.2}{XY} \to XY = \frac{20\sin 68.2}{\sin 43.6} \approx 26.9$

$XY = XZ \approx 26.9$ cm

3



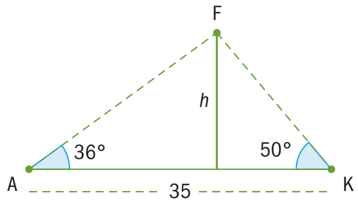
$A\hat{B}T = 180 - 75 = 105°$

$A\hat{T}B = 180 - 40 - 105 = 35°$

$\frac{\sin 35}{2} = \frac{\sin 105}{AT} \to AT = \frac{2\sin 105}{\sin 35} \approx 3.37$ km

$\frac{\sin 35}{2} = \frac{\sin 40}{BT} \to BT = \frac{2\sin 40}{\sin 35} \approx 2.24$ km

4



$A\hat{F}K = 180 - 50 - 36 = 94°$

$\frac{\sin 94}{35} = \frac{\sin 36}{FK} \to FK = \frac{35\sin 36}{\sin 94} \approx 20.6227$

$\sin 50 = \frac{h}{FK} \to h = FK\sin 50 \approx 15.8$ m

Investigation – ambiguous triangles

1 $\frac{\sin 32}{3} = \frac{\sin C}{5} \to \sin C = \frac{5\sin 32}{3}$

$\to \hat{C}_1 = \sin^{-1}\left(\frac{5\sin 32}{3}\right) \approx 62.0°$

$\hat{C}_2 = 180 - \hat{C}_1 \approx 118°$

The angles are supplementary.

2 $\hat{B}_1 = 180 - 32 - \hat{C}_1 \approx 86.0°$

$\frac{\sin 32}{3} = \frac{\sin B_1}{AC} \to AC = \frac{3\sin B_1}{\sin 32} \approx 5.65\text{ cm}$

$\hat{B}_2 = 180 - 32 - \hat{C}_2 \approx 30.0°$

$\frac{\sin 32}{3} = \frac{\sin B_2}{AC} \to AC = \frac{3\sin B_2}{\sin 32} \approx 2.83\text{ cm}$

Exercise 11H

1 a $\frac{\sin 30}{4} = \frac{\sin C_1}{7} \to \sin C_1 = \frac{7\sin 30}{4}$

$\to C_1 = \sin^{-1}\left(\frac{7\sin 30}{4}\right) \approx 61.0°$

$\hat{B}_1 = 180 - 30 - \hat{C}_1 \approx 89.0°$

$\frac{\sin 30}{4} = \frac{\sin B_1}{b_1} \to b_1 = \frac{4\sin B_1}{\sin 30} \approx 8.0\text{ cm}$

$C_2 = 180 - C_1 \approx 119.0°$

$\hat{B}_2 = 180 - 30 - \hat{C}_2 \approx 31.0°$

$\frac{\sin 30}{4} = \frac{\sin B_2}{b_2} \to b_2 = \frac{4\sin B_2}{\sin 30} \approx 4.1\text{ cm}$

b $\frac{\sin 50}{17} = \frac{\sin C_1}{21} \to \sin C_1 = \frac{21\sin 50}{17}$

$\to C_1 = \sin^{-1}\left(\frac{21\sin 50}{17}\right) \approx 71.1°$

$\hat{A}_1 = 180 - 50 - \hat{C}_1 \approx 58.9°$

$\frac{\sin 50}{17} = \frac{\sin A_1}{a_1} \to a_1 = \frac{17\sin A_1}{\sin 50} \approx 19.0\text{ cm}$

$C_2 = 180 - C_1 \approx 108.9°$

$\hat{A}_2 = 180 - 50 - \hat{C}_2 \approx 21.1°$

$\frac{\sin 50}{17} = \frac{\sin A_2}{a_2} \to a_2 = \frac{17\sin A_2}{\sin 50} \approx 8.0\text{ cm}$

c $\frac{\sin 20}{2.5} = \frac{\sin B_1}{6.8} \to \sin B_1 = \frac{6.8\sin 20}{2.5}$

$\to B_1 = \sin^{-1}\left(\frac{6.8\sin 20}{2.5}\right) \approx 68.5°$

$A_1 = 180 - 20 - B_1 \approx 91.5°$

$\frac{\sin 20}{2.5} = \frac{\sin A_1}{a_1} \to a_1 = \frac{2.5\sin A_1}{\sin 20} \approx 7.3\text{ cm}$

$B_2 = 180 - B_1 \approx 111.5°$

$A_2 = 180 - 20 - B_2 \approx 48.5°$

$\frac{\sin 20}{2.5} = \frac{\sin A_2}{a_2} \to a_2 = \frac{2.5\sin A_2}{\sin 20} \approx 5.5\text{ cm}$

d $\frac{\sin 42}{33} = \frac{\sin C}{25} \to \sin C = \frac{25\sin 42}{33}$

$\to C = \sin^{-1}\left(\frac{25\sin 42}{33}\right) \approx 30.5°$

$B_1 = 180 - 42 - 30.5 \approx 107.5°$

$\frac{\sin 42}{33} = \frac{\sin B_1}{b_1} \to b_1 = \frac{33\sin B_1}{\sin 42} \approx 47.0\text{ cm}$

e $\frac{\sin 70}{25} = \frac{\sin B}{28} \to \sin B \approx 1.05 \to$ *triangle does not exist*

f $\frac{\sin 70}{25} = \frac{\sin B_1}{26} \to \sin B_1 = \frac{26\sin 70}{25}$

$\to B_1 = \sin^{-1}\left(\frac{26\sin 70}{25}\right) \approx 77.8°$

$C_1 = 180 - 70 - B_1 \approx 32.2°$

$\frac{\sin 70}{25} = \frac{\sin C_1}{c_1} \to c_1 = \frac{25\sin C_1}{\sin 70} \approx 14.2\text{ cm}$

$B_2 = 180 - B_1 \approx 102.2°$

$C_2 = 180 - 70 - B_2 \approx 7.8°$

$\frac{\sin 70}{25} = \frac{\sin C_2}{c_2} \to c_2 = \frac{25\sin C_2}{\sin 70} \approx 3.6\text{ cm}$

g $\frac{\sin 45}{22} = \frac{\sin B}{14} \to \sin B_1 = \frac{14\sin 45}{22}$

$\to B_1 = \sin^{-1}\left(\frac{14\sin 45}{22}\right) \approx 26.7°$

$C = 180 - 45 - B \approx 108.3°$

$\frac{\sin 45}{22} = \frac{\sin C}{c} \to c = \frac{22\sin C}{\sin 45} \approx 29.5\text{ cm}$

h $\frac{\sin 56}{45} = \frac{\sin C_1}{50} \to \sin C_1 = \frac{50\sin 56}{45}$

$\to C_1 = \sin^{-1}\left(\frac{50\sin 56}{45}\right) \approx 67.1°$

$A_1 = 180 - 56 - C_1 \approx 56.9°$

$\frac{\sin 56}{45} = \frac{\sin A_1}{a_1} \to a_1 = \frac{45\sin A_1}{\sin 56} \approx 45.5\text{ cm}$

$C_2 = 180 - C_1 \approx 112.9°$

$A_2 = 180 - 56 - C_2 \approx 11.1°$

$\frac{\sin 56}{45} = \frac{\sin A_2}{a_2} \to a_2 = \frac{45\sin A_2}{\sin 56} \approx 10.4\text{ cm}$

2 a $BE^2 + 6^2 = 10^2 \to BE = \sqrt{10^2 - 6^2} = 8\text{ m}$

$CE^2 + 8^2 = 10^2 \to CE = \sqrt{10^2 - 8^2} = 6\text{ m}$

$DE^2 + 8^2 = 17^2 \to DE = \sqrt{17^2 - 8^2} = 15\text{ m}$

b $AEB = 90°$

$\cos EAB = \frac{6}{10} \to EAB = \cos^{-1}\left(\frac{6}{10}\right) \approx 53.1°$

$BCE = EAB \approx 53.1°$

$\sin BCE = \frac{8}{10} \to BCE = \sin^{-1}\left(\frac{8}{10}\right) \approx 53.1°$

$BCD = 180 - BDC \approx 126.9°$

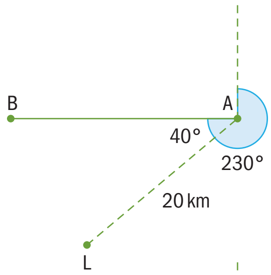
$\frac{\sin BCD}{17} = \frac{\sin BDC}{10} \to BDC = \sin^{-1}\left(\frac{10\sin BCD}{17}\right) \approx 28.1°$

$ABD = 180 - EAB - BDC \approx 98.8°$

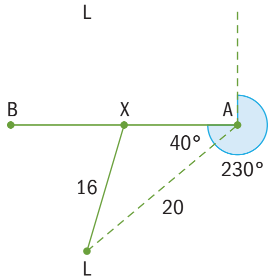
$CBD = 180 - BCD - BDC \approx 25.1°$

c Given side $BD = 17$ m in triangle ABD and angle $\hat{D} = 28.1°$, and side $AB = 10$, then there are 2 possible triangles fitting this data, namely DBA and DBC.

3 a



b



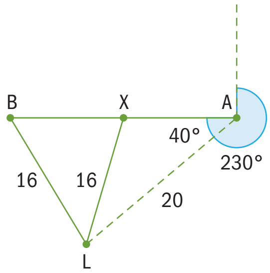
$\frac{\sin 40}{16} = \frac{\sin AXL}{20} \to AXL = \sin^{-1}\left(\frac{20\sin 40}{16}\right)$

$\approx 180 - 53.464 \approx 126.536°$

$$ALX = 180 - 40 - AXL \approx 13.464°$$

$$\frac{\sin 40}{16} = \frac{\sin ALX}{AX} \rightarrow AX = \frac{16 \sin ALX}{\sin 40} \approx 5.80 \text{ km}$$

c



$$\frac{\sin 40}{16} = \frac{\sin ABL}{20} \rightarrow$$

$$ABL = \sin^{-1}\left(\frac{20 \sin 40}{16}\right) \approx 53.464°$$

$$BLX = 180 - 2(ABL) \approx 73.07°$$

$$\frac{\sin ABL}{16} = \frac{\sin BLX}{BX} \rightarrow BX = \frac{16 \sin BLX}{\sin ABL} \approx 19.1 \text{ km}$$

d bearing = 90 + ABL ≈ 143.5°

Exercise 11I

1 a $a^2 = 43^2 + 72^2 - 2(43)(72)\cos 64$

$$\rightarrow a = \sqrt{43^2 + 72^2 - 2(43)(72)\cos 64}$$
$$\approx 65.7 \text{ m}$$

$$\frac{\sin 64}{a} = \frac{\sin B}{43} \rightarrow B = \sin^{-1}\left(\frac{43 \sin 64}{a}\right) \approx 36.0°$$

$$C = 180 - 64 - B \approx 80.0°$$

b $\cos A = \frac{33^2 + 41^2 - 20^2}{2(33)(41)}$

$$\rightarrow A = \cos^{-1}\left(\frac{33^2 + 41^2 - 20^2}{2(33)(41)}\right) \approx 28.9°$$

$$\cos B = \frac{20^2 + 41^2 - 33^2}{2(20)(41)} \rightarrow$$

$$B = \cos^{-1}\left(\frac{20^2 + 41^2 - 33^2}{2(20)(41)}\right) \approx 52.8°$$

$$C = 180 - A - B \approx 98.4°$$

c $\cos A = \frac{4.9^2 + 2.4^2 - 3.6^2}{2(4.9)(2.4)}$

$$\rightarrow A = \cos^{-1}\left(\frac{4.9^2 + 2.4^2 - 3.6^2}{2(4.9)(2.4)}\right) \approx 44.4°$$

$$\cos B = \frac{3.6^2 + 2.4^2 - 4.9^2}{2(3.6)(2.4)}$$

$$\rightarrow B = \cos^{-1}\left(\frac{3.6^2 + 2.4^2 - 4.9^2}{2(3.6)(2.4)}\right) \approx 107.8°$$

$$C = 180 - A - B \approx 27.8°$$

d $b^2 = 10^2 + 14^2 - 2(10)(14)\cos 31$

$$\rightarrow b = \sqrt{10^2 + 14^2 - 2(10)(14)\cos 31}$$
$$\approx 7.5 \text{ m}$$

$$\frac{\sin 31}{b} = \frac{\sin A}{10} \rightarrow A = \sin^{-1}\left(\frac{10 \sin 31}{b}\right) \approx 43.5°$$

$$C = 180 - 31 - A \approx 105.5°$$

e $c^2 = 75^2 + 86^2 - 2(75)(86)\cos 70$

$$\rightarrow c = \sqrt{75^2 + 86^2 - 2(75)(86)\cos 70}$$
$$\approx 92.8 \text{ m}$$

$$\frac{\sin 70}{c} = \frac{\sin A}{75} \rightarrow A = \sin^{-1}\left(\frac{75 \sin 70}{c}\right) \approx 49.4°$$

$$B = 180 - 70 - A \approx 60.6°$$

f $\cos A = \frac{50^2 + 58^2 - 45^2}{2(50)(58)}$

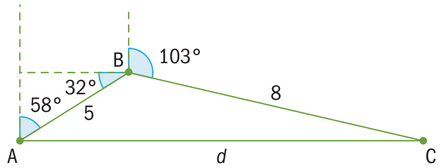
$$\rightarrow A = \cos^{-1}\left(\frac{50^2 + 58^2 - 45^2}{2(50)(58)}\right) \approx 48.6°$$

$$\cos B = \frac{45^2 + 58^2 - 50^2}{2(45)(58)}$$

$$\rightarrow B = \cos^{-1}\left(\frac{45^2 + 58^2 - 50^2}{2(45)(58)}\right) \approx 56.4°$$

$$C = 180 - A - B \approx 75.0°$$

2

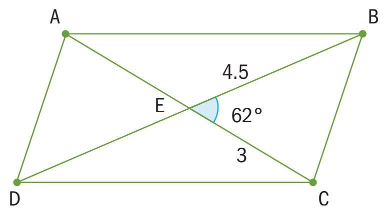


$$A\hat{B}C = 270 - 32 - 103 = 135°$$

$$d^2 = 5^2 + 8^2 - 2(5)(8)\cos 135$$

$$\rightarrow d = \sqrt{5^2 + 8^2 - 2(5)(8)\cos 135} \approx 12.1 \text{ km}$$

3

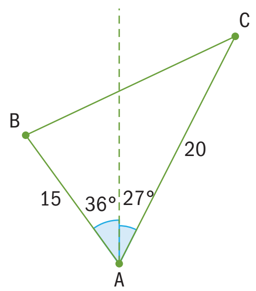


$$AD = BC = \sqrt{4.5^2 + 3^2 - 2(4.5)(3)\cos 62} \approx 4.07 \text{ cm}$$

$$A\hat{E}B = 180 - 62 = 118°$$

$$AB = CD = \sqrt{4.5^2 + 3^2 - 2(4.5)(3)\cos 118} \approx 6.48 \text{ cm}$$

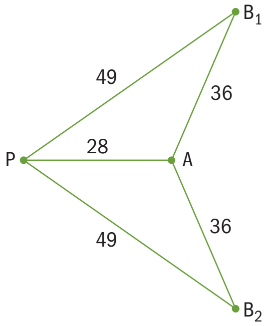
4



$$B\hat{A}C = 36 + 27 = 63°$$

$$BC = \sqrt{15^2 + 20^2 - 2(15)(20)\cos 63} \approx 18.8 \text{ km}$$

5

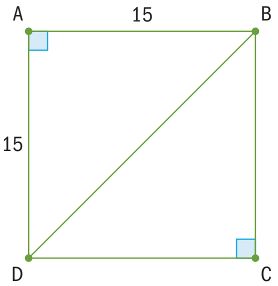


$$\cos A\hat{P}B = \frac{49^2 + 28^2 - 36^2}{2(49)(28)}$$

$$\rightarrow A\hat{P}B = \cos^{-1}\left(\frac{49^2 + 28^2 - 36^2}{2(49)(28)}\right) \approx 46.5°$$

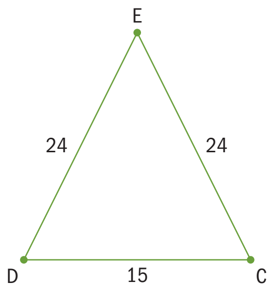
E46.5°N or E46.5°S which is a bearing of 043.5° or 136.5° (since 90 − 46.5 = 43.5 and 90 + 46.5 = 136.5).

6 a



Triangle *ABD* is an isosceles right triangle, so angle *ABD* = 45°

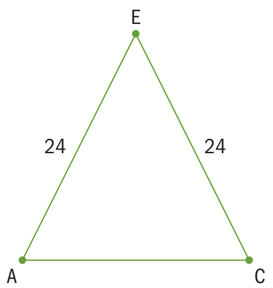
b



$$\cos E\hat{D}C = \frac{24^2 + 15^2 - 24^2}{2(24)(15)}$$

$$\rightarrow E\hat{D}C = \cos^{-1}\left(\frac{24^2 + 15^2 - 24^2}{2(24)(15)}\right) \approx 71.8°$$

c



$$AC = \sqrt{15^2 + 15^2} = 15\sqrt{2}$$

$$\cos E\hat{A}C = \frac{24^2 + \left(15\sqrt{2}\right)^2 - 24^2}{2(24)\left(15\sqrt{2}\right)}$$

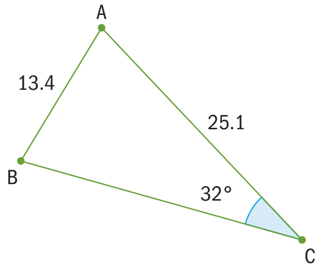
$$E\hat{A}C = \cos^{-1}\left(\frac{24^2 + \left(15\sqrt{2}\right)^2 - 24^2}{2(24)\left(15\sqrt{2}\right)}\right) \approx 63.8°$$

Exercise 11J

1 a $Area = \frac{1}{2}(6.8)(9.4)\sin 56.5 \approx 26.7 \text{ cm}^2$

b $Area = \frac{1}{2}(10)(9)\sin 115 \approx 40.8 \text{ cm}^2$

c

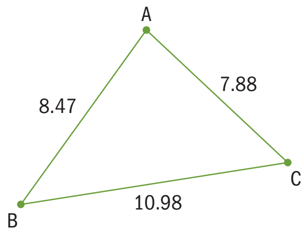


$$\frac{\sin 32}{13.4} = \frac{\sin B}{25.1} \quad \rightarrow B = \sin^{-1}\left(\frac{25.1 \sin 32}{13.4}\right) \approx 83.03°$$

$$A = 180 - 32 - B \approx 64.97°$$

$$Area = \frac{1}{2}(13.4)(25.1)\sin A \approx 152 \text{ cm}^2$$

d



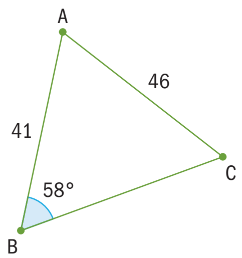
$$\cos A = \frac{8.74^2 + (7.88)^2 - 10.98^2}{2(8.74)(7.88)}$$

$$\rightarrow \hat{A} = \cos^{-1}\left(\frac{8.74^2 + (7.88)^2 - 10.98^2}{2(8.74)(7.88)}\right)$$

$$\approx 82.524°$$

$$Area = \frac{1}{2}(8.74)(7.88)\sin A \approx 34.1 \text{ cm}^2$$

e

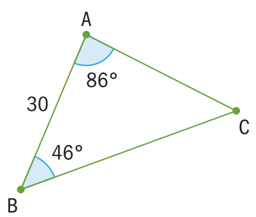


$$\frac{\sin 58}{46} = \frac{\sin C}{41} \quad \rightarrow C = \sin^{-1}\left(\frac{41 \sin 58}{46}\right) \approx 49.10°$$

$$A = 180 - 58 - C \approx 72.899°$$

$$Area = \frac{1}{2}(41)(46)\sin A \approx 901 \text{ cm}^2$$

f



$$C = 180 - 86 - 46 = 48°$$

$$\frac{\sin 48}{30} = \frac{\sin 86}{BC} \quad \rightarrow BC = \frac{30 \sin 86}{\sin 48} \approx 40.27$$

$$Area = \frac{1}{2}(30)(BC)\sin 46 \approx 435 \text{ cm}^2$$

2 $\frac{1}{2}(15)(18)\sin\theta \approx 100$

$\rightarrow \theta = \sin^{-1}\left(\frac{100}{\left(\frac{1}{2}(15)(18)\right)}\right) \approx 47.8°$

3 $\frac{1}{2}(x)(33.9)\sin 57.4 \approx 324$

$\rightarrow x = \frac{324}{\left(\frac{1}{2}(33.9)\sin 57.4\right)} \approx 22.7 \text{ cm}$

4 **a** $\cos A = \frac{16.4^2 + 10.2^2 - 17.2^2}{2(16.4)(10.2)}$

$\rightarrow \hat{A} = \cos^{-1}\left(\frac{16.4^2 + 10.2^2 - 17.2^2}{2(16.4)(10.2)}\right)$

$\approx 76.7°$

b $Area = \frac{1}{2}(16.4)(10.2)\sin A \approx 81.4 \text{ cm}^2$

5 $\frac{1}{2}(2x+3)(4x+5)\sin 30 = 30$

$\frac{1}{2}(8x^2 + 22x + 15)\left(\frac{1}{2}\right) = 30 \rightarrow 8x^2 + 22x + 15 = 120$

$\rightarrow 8x^2 + 22x - 105 = 0$

$x = 2.5$

6 $\frac{1}{2}(8)(11)\sin A \approx 20$

$\rightarrow A = \sin^{-1}\left(\frac{20}{44}\right) \approx 27.0357° \text{ or } 152.9643°$

$x^2 = 11^2 + 8^2 - 2(11)(8)\cos 27.0357$

$\rightarrow x = \sqrt{11^2 + 8^2 - 2(11)(8)\cos 27.0357}$

$\approx 5.31 \text{ mm}$

$x^2 = 11^2 + 8^2 - 2(11)(8)\cos 152.9643$

$\rightarrow x = \sqrt{11^2 + 8^2 - 2(11)(8)\cos 152.9643}$

$\approx 18.5 \text{ mm}$

Exercise 11K

1 $1.7(5.6) = 9.52 \text{ cm}$

2 $3.25(12) = 39 \text{ cm}$

3 $\theta(2.5) = 12.5 \Rightarrow \theta = 5 \text{ rad}$

4 $\text{area} = \frac{1}{2}(2.4)(50^2) = 3000 \text{ cm}^2$

perimeter = radius + radius + arc length

$\text{perimeter} = 50 + 50 + 2.4(50) = 220 \text{ cm}$

5 $\text{area} = \frac{1}{2}(5.1)(3^2) = 22.95 \text{ cm}^2$

$\text{perimeter} = 3 + 3 + 5.1(3) = 21.3 \text{ cm}$

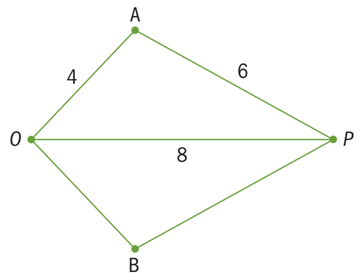
6 $\theta(r) = 27.2 \quad \rightarrow \theta = \frac{27.2}{r}$

$\frac{1}{2}\theta(r^2) = 217.6 \quad \rightarrow \theta = \frac{435.2}{r^2}$

$\frac{27.2}{r} = \frac{435.2}{r^2} \quad \rightarrow 27.2r = 435.2 \rightarrow r = 16 \text{ cm}$

$\theta = \frac{27.2}{16} = 1.7 \text{ rad}$

7



$\cos A\hat{O}P = \frac{4^2 + (8)^2 - 6^2}{2(4)(8)} \quad \rightarrow A\hat{O}P = \cos^{-1}\left(\frac{4^2 + (8)^2 - 6^2}{2(4)(8)}\right)$

$\approx 0.812756 \text{ rad}$

$O = 2A\hat{O}P \approx 1.6255 \text{ rad}$

$\cos O\hat{P}A = \frac{6^2 + (8)^2 - 4^2}{2(6)(8)} \rightarrow O\hat{P}A = \cos^{-1}\left(\frac{6^2 + (8)^2 - 4^2}{2(6)(8)}\right)$

$\approx 0.50536 \text{ rad}$

$P = 2O\hat{P}A \approx 1.01072 \text{ rad}$

$\text{area of quadrilateral } OAPB = 2\left(\frac{1}{2}(6)(8)\sin O\hat{P}A\right)$

≈ 23.2379

$\text{area of sector } AOB \approx \frac{1}{2}(1.6255)(4^2) \approx 13.004$

$\text{area of sector } APB \approx \frac{1}{2}(1.01072)(6^2) \approx 18.19296$

overlapping area = area of sector AOB
+ area of sector APB
– area of quadrilateral OAPB

$\text{overlapping area} \approx 13.004 + 18.19296 - 23.2379$

$\approx 7.96 \text{ cm}^2$

Exercise 11L

1 **a** $75\left(\frac{\pi}{180}\right) = \frac{5\pi}{12}$

b $240\left(\frac{\pi}{180}\right) = \frac{4\pi}{3}$

c $80\left(\frac{\pi}{180}\right) = \frac{4\pi}{9}$

d $330\left(\frac{\pi}{180}\right) = \frac{11\pi}{6}$

2 **a** $56\left(\frac{\pi}{180}\right) \approx 0.977 \text{ rad}$

b $107\left(\frac{\pi}{180}\right) \approx 1.87 \text{ rad}$

c $324\left(\frac{\pi}{180}\right) \approx 5.65 \text{ rad}$

d $230\left(\frac{\pi}{180}\right) \approx 4.01 \text{ rad}$

3 **a** $\frac{5\pi}{6}\left(\frac{180}{\pi}\right) = 150°$

b $\frac{5\pi}{3}\left(\frac{180}{\pi}\right) = 300°$

c $\frac{3\pi}{2}\left(\frac{180}{\pi}\right) = 270°$

d $\frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$

4 **a** $1.5\left(\frac{180}{\pi}\right) \approx 85.9°$

b $0.36\left(\frac{180}{\pi}\right) \approx 20.6°$

c $2.38\left(\frac{180}{\pi}\right) \approx 136°$

d $3.59\left(\frac{180}{\pi}\right) \approx 206°$

Exercise 11M

1 **a** $\sin\frac{\pi}{4} = \sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

b $\cos\frac{2\pi}{3} = \cos 120° = -\frac{1}{2}$

c $\tan\frac{\pi}{6} = \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

d $\sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$

2 **a** 0.892

b 0.949

c –1.12

d 0.667

These values are found by using your GDC in radians mode.

3 **a** $\frac{1}{2}(4.5)(4.5)\sin 1.3 \approx 9.76 \text{ cm}^2$

b $BC = \sqrt{4.5^2 + 4.5^2 - 2(4.5)(4.5)\cos 1.3} \approx 5.45 \text{ cm}$

Shaded area = area of circle − area of sector

$= \pi(4.5)^2 - \frac{1.3(4.5)^2}{2}$

$= 50.5 \text{ cm}^2$ (3 sf)

4 **a** $\frac{\sin 0.94}{11} = \frac{\sin B}{3} \rightarrow B = \sin^{-1}\left(\frac{3\sin 0.94}{11}\right) \approx 0.222 \text{ rad}$

$A = \pi - 0.94 - B \approx 1.9795 \text{ rad}$

Area of $\Delta OAB = \frac{1}{2}(3)(11)\sin A \approx 15.141$

Area of sector $= \frac{1}{2}(0.94)(3^2) = 4.23$

shaded area ≈ 15.141 − 4.23 ≈ 10.9 m²

5 **a** Area of $\Delta POQ = \frac{1}{2}(6)(6)\sin 1.25 \approx 17.1 \text{ cm}^2$

b $Q\hat{O}R = \cos^{-1}\left(\frac{6^2 + 6^2 - 11.2^2}{2(6)(6)}\right) \approx 2.407 \text{ rad}$

Area of $\Delta QOR = \frac{1}{2}(6)(6)\sin Q\hat{O}R \approx 12.1 \text{ cm}^2$

c $\theta = 2\pi - 1.25 - Q\hat{O}R \approx 2.63 \text{ rad}$

d $\theta(6) \approx 15.8 \text{ cm}$

Review exercise

1 $\sin 45 = \frac{1}{\sqrt{2}} = \frac{7}{AB} \rightarrow AB = 7\sqrt{2} \text{ cm}$

2 **a** $\sin X\hat{Z}Y = \frac{8}{16} = \frac{1}{2} \rightarrow X\hat{Z}Y = 30°$

b $\tan X\hat{Z}Y = \tan 30 = \frac{1}{\sqrt{3}} = \frac{8}{YZ}$

$\rightarrow YZ = 8\sqrt{3} \text{ cm}$

3 $\tan\theta = \frac{2}{5}$

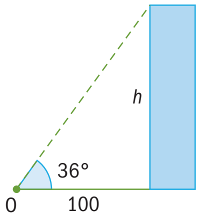
4 $\frac{1}{2}(4)(10)\sin 30 = \frac{1}{2}(4)(10)\left(\frac{1}{2}\right) = 10 \text{ cm}^2$

5 **a** $2.5(10) = 25 \text{ cm}$

b $\frac{1}{2}(2.5)(10^2) = \frac{1}{2}(2.5)(100) = 50(2.5) = 125 \text{ cm}^2$

Review exercise

1



$\tan 36 = \frac{h}{100} \rightarrow h = 100\tan 36 \approx 72.7 \text{ m}$

2 **a** $(\cos 32, \sin 32) \rightarrow (0.848, 0.530)$

b $C\left(\cos A\hat{O}C, \sin A\hat{O}C\right)$

$\rightarrow A\hat{O}C = \cos^{-1} 0.294 \approx 72.9°$

c $A\hat{O}D = A\hat{O}C + 54 \approx 126.96°$

$D\left(\cos A\hat{O}D, \sin A\hat{O}D\right) = (-0.600, 0.800)$

3 **a** $\hat{Y} = 180 - 42.4 - 82.9 = 54.7°$

b $\frac{\sin 82.9}{13.2} = \frac{\sin 54.7}{XZ} \rightarrow XZ = \frac{13.2\sin 54.7}{\sin 82.9} \approx 10.9 \text{ cm}$

4 **a** $PR = \sqrt{9.5^2 + 11.5^2 - 2(9.5)(11.5)\cos 118} \approx 18.03 \text{ m}$

b $\frac{\sin 118}{PR} = \frac{\sin\hat{P}}{11.5} \rightarrow \hat{P} = \sin^{-1}\left(\frac{11.5\sin 118}{PR}\right) \approx 34.3°$

5 **a** $\frac{1}{2}(4)(5.83)\sin\hat{C} = 10$

$\rightarrow \hat{C} = \sin^{-1}\left(\frac{10}{\left(\frac{1}{2}(4)(5.83)\right)}\right) \approx 121°$

b $AB = \sqrt{4^2 + 5.83^2 - 2(4)(5.83)\cos\hat{C}} \approx 8.60 \text{ cm}$

6 **a** $A\hat{P}B = 170 - 50 = 120°$

$AB = \sqrt{24^2 + 38^2 - 2(24)(38)\cos 120} \approx 54.1 \text{ km}$

7 **a** $\frac{\sin 82}{15} = \frac{\sin x}{8} \rightarrow x = \sin^{-1}\left(\frac{8\sin 82}{15}\right) \approx 31.9°$

b $A\hat{D}C = 180 - 82 - x \approx 66.12°$

$\frac{\sin 82}{15} = \frac{\sin A\hat{D}C}{AC} \rightarrow AC = \frac{15\sin A\hat{D}C}{\sin 82} \approx 13.9 \text{ cm}$

c $y = \cos^{-1}\left(\frac{7^2 + 9^2 - AC^2}{2(7)(9)}\right) \approx 119°$

d $\frac{1}{2}(7)(9)\sin y \approx 27.6 \text{ cm}^2$

8 **a** $\hat{B} = \pi - 1.75 - 0.93 \approx 0.4616 \text{ rad}$

$\frac{\sin\hat{B}}{12} = \frac{\sin 0.93}{BC} \rightarrow BC = \frac{12\sin 0.93}{\sin\hat{B}} \approx 21.6 \text{ cm}$

b $\frac{\sin 0.93}{BC} = \frac{\sin 1.75}{AB} \rightarrow AB = \frac{BC\sin 1.75}{\sin 0.93} \approx 26.512$

$DB = AB - 12 \approx 14.5 \text{ cm}$

c $0.93(12) = 11.16 \approx 11.2 \text{ cm}$

d $DB + \text{arc}DEC + BC \approx 47.3 \text{ cm}$

12 Vectors

Skills check

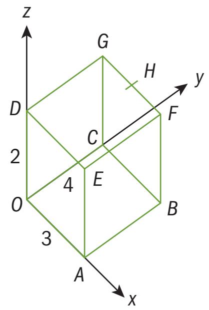
1 **a** $A = (3, 0, 0)$

b $B = (3, 4, 0)$

c $E = (3, 0, 2)$

d $F = (3, 4, 2)$

e $H = \left(\frac{3}{2}, 4, 2\right)$



2 $x^2 = 3^2 + 6^2$

$= 9 + 36$

$= 45$

$x = \sqrt{45} \approx 6.71$

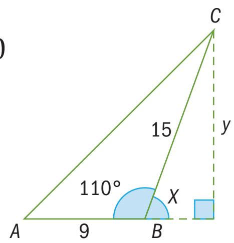
3 **a** $X = 180 - 110 = 70$

$\cos X = \frac{z}{15} \Rightarrow z = 15\cos 70$

≈ 5.13

$\sin X = \frac{y}{15} \Rightarrow y = 15\sin 70$

≈ 14.1



$(AC)^2 = y^2 + (9+z)^2$

$= (14.1)^2 + (9+5.13)^2$

$AC = \sqrt{432.5}$

$= 20.8$

$= 21\,\text{cm}\,(\text{to the nearest centimetre})$

b Using the Cosine Rule

$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos(A\hat{B}C)$

$(9.7)^2 = (8.6)^2 + (3.1)^2 - 2(8.6)(3.1)\cos(A\hat{B}C)$

$A\hat{B}C = \cos^{-1}\left[\frac{(8.6)^2 + (3.1)^2 - (9.7)^2}{2(8.6)(3.1)}\right]$

$\approx 101.4°$

Exercise 12A

1 **a** $\mathbf{x} = -2\mathbf{i} + 3\mathbf{j}$

b $\mathbf{y} = 7\mathbf{j}$

c $\mathbf{z} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

2 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

b $\overrightarrow{CD} = \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix}$

c $\overrightarrow{EF} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3 **a** $= -3\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

b $= -2\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

c $= 3\mathbf{i} + 8\mathbf{j} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

d $= 6\mathbf{j} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

e $= -3\mathbf{i} - 6\mathbf{j} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

4 **a** $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

b $\left|\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \approx 3.16$

c $|2\mathbf{i} + 5\mathbf{j}| = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.39$

d $\left|\begin{pmatrix} 2.8 \\ 4.5 \end{pmatrix}\right| = \sqrt{(2.8)^2 + (4.5)^2} = 5.3$

e $|2\mathbf{i} - 5\mathbf{j}| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.39$

5 **a** $\left|\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38} \approx 6.16$

b $\left|\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}\right| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \approx 5.10$

c $|2\mathbf{i} + 2\mathbf{j} + \mathbf{k}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

d $\left|\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}\right| = \sqrt{(-3)^2 + 2^2 + 6^2} = \sqrt{49} = 7$

e $|\mathbf{j} - \mathbf{k}| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \approx 1.41$

Exercise 12B

1 **a** $= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

c $= \begin{pmatrix} -6 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -3\mathbf{b}$

d $= \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{2}\mathbf{a}$

$\mathbf{e} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = -5\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -5\mathbf{b}$

$\mathbf{f} = \begin{pmatrix} -4 \\ -8 \end{pmatrix}$

We must have s & t so that

$-5 = 2s + 2t \quad (1)$

$-8 = 4s - t \quad (2)$

$2 \times (2): -16 = 8s - 2t \quad (3)$

$(1) + (3): -21 = 10s$

$s = \frac{-21}{10}$

from $(2): -8 = \frac{-84}{10} - t$

$t = 8 - \frac{84}{10}$

$= \frac{-2}{5}$

so $\mathbf{f} = -2\begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2\mathbf{a}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

2 $\mathbf{a} = \begin{pmatrix} 0.1 \\ 0.7 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

$= \frac{1}{10}(\mathbf{i} + 7\mathbf{j})$

$\mathbf{a}$ is parallel to $\mathbf{i} + 7\mathbf{j}$ with $\frac{1}{10}$ the magnitude.

$\mathbf{b} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$

$= -1(\mathbf{i} + 7\mathbf{j})$

$\mathbf{b}$ is parallel to $(\mathbf{i} + 7\mathbf{j})$ with opposite direction.

$\mathbf{c} = \begin{pmatrix} -0.05 \\ -0.03 \end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{d} = \begin{pmatrix} -10 \\ 70 \end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{e} = 60\mathbf{i} + 420\mathbf{j}$

$= 60(\mathbf{i} + 7\mathbf{j})$

$\mathbf{e}$ is parallel to $(\mathbf{i} + 7\mathbf{j})$ with 60 times the magnitude

$\mathbf{f} = (6\mathbf{i} - 42\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{g} = (-\mathbf{i} + 7\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

3 **a** For parallel vectors, $\mathbf{r} = k\,\mathbf{s}$ for some k

$(4\mathbf{i} + t\,\mathbf{j}) = k(14\mathbf{i} - 12\mathbf{j})$

$4 = 14k$

$k = \frac{4}{14}$

$= \frac{2}{7}$

$t = -12k$

$= -12 \times \frac{2}{7}$

$= \frac{-24}{7}$

b For parallel vectors, $\mathbf{a} = k\,\mathbf{b}$ for some k

$\begin{pmatrix} t \\ -8 \end{pmatrix} = k\begin{pmatrix} 7 \\ -10 \end{pmatrix}$

$-8 = -10k$

$k = \frac{-8}{-10}$

$= \frac{4}{5}.$

so $t = 7k$

$= 7\left(\frac{4}{5}\right)$

$= \frac{28}{5}$

4 For parallel vectors, $\mathbf{v} = k\,\mathbf{w}$ for some k

$t\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = k(5\mathbf{i} + \mathbf{j} + s\,\mathbf{k})$

$-5 = k$

so $t = 5k$

$t = 5(-5)$

$= -25$

$8 = sk = (-5)s$

$s = \frac{-8}{5}$

5 **a** $\overrightarrow{OG} = \mathbf{j} + \mathbf{k}$

b $\overrightarrow{BD} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

c $\overrightarrow{AD} = -\mathbf{i} + \mathbf{k}$

d $\overrightarrow{OM} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

6 **a** $\overrightarrow{OG} = 4\mathbf{j} + 3\mathbf{k}$

b $\overrightarrow{BD} = -5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

c $\overrightarrow{AD} = -5\mathbf{i} + 3\mathbf{k}$

d $\overrightarrow{OM} = \frac{5}{2}\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

Exercise 12C

1 $\overrightarrow{OP}=\begin{pmatrix}7\\4\end{pmatrix}$ $\quad\overrightarrow{OQ}=\begin{pmatrix}2\\3\end{pmatrix}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}7\\4\end{pmatrix}=\begin{pmatrix}-5\\-1\end{pmatrix}$$

$$\overrightarrow{QP}=-\overrightarrow{PQ}=\begin{pmatrix}5\\1\end{pmatrix}$$

2 $A=\begin{pmatrix}5\\1\end{pmatrix}$ $B=\begin{pmatrix}1\\-3\end{pmatrix}$ $C=\begin{pmatrix}-2\\4\end{pmatrix}$

a $\overrightarrow{AB}=B-A=\begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}-4\\-4\end{pmatrix}$

b $\overrightarrow{BA}=-\overrightarrow{AB}=\begin{pmatrix}4\\4\end{pmatrix}$

c $\overrightarrow{AC}=C-A=\begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}-7\\3\end{pmatrix}$

d $\overrightarrow{CB}=B-C=\begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}3\\-7\end{pmatrix}$

3 a $\overrightarrow{OP}=\begin{pmatrix}2\\-3\\5\end{pmatrix}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$

b vector is $-\begin{pmatrix}1\\-5\\6\end{pmatrix}=-\mathbf{i}+5\mathbf{j}-6\mathbf{k}$

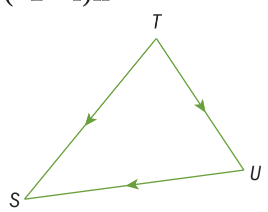
c vector is $\begin{pmatrix}1\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\-3\\5\end{pmatrix}=\begin{pmatrix}-1\\5\\-6\end{pmatrix}=-\mathbf{i}+5\mathbf{j}-6\mathbf{k}$

d vector is $\begin{pmatrix}2\\-3\\5\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\-5\\6\end{pmatrix}=\mathbf{i}-5\mathbf{j}+6\mathbf{k}$

4 $\overrightarrow{LM}=\overrightarrow{LN}+\overrightarrow{NM}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\begin{pmatrix}4\\-2\\-3\end{pmatrix}=\begin{pmatrix}5\\-4\\-3\end{pmatrix}$

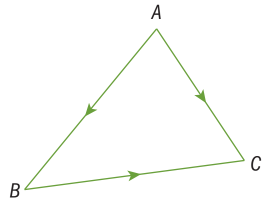
5 From the diagram, we see

$$\begin{aligned}\overrightarrow{US}&=-\overrightarrow{TU}+\overrightarrow{TS}\\&=-(\mathbf{i}-4\mathbf{j}+2\mathbf{k})+(3\mathbf{i}+4\mathbf{j}-\mathbf{k})\\&=(-1+3)\mathbf{i}+(4+4)\mathbf{j}+(-2-1)\mathbf{k}\\&=2\mathbf{i}+8\mathbf{j}-3\mathbf{k}\end{aligned}$$



6 From the diagram,

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=0$



$$\begin{pmatrix}1\\y\\-2\end{pmatrix}+\begin{pmatrix}2x\\-3\\z\end{pmatrix}-\begin{pmatrix}1\\4\\x+y\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$1+2x-1=0\Rightarrow 0+2x=0$ (1)

$y-3-4=0\Rightarrow y-7=0$ (2)

$-2+z-(x+y)\Rightarrow -x-y+z-2=0$ (3)

(1) $\Rightarrow x=0$

(2) $\Rightarrow y=7$

(3) $\Rightarrow -2+z-7=0$

$z=9$

Exercise 12D

1 $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(-2\mathbf{i}+3\mathbf{j}-\mathbf{k})-(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

$=(-2-1)\mathbf{i}+(3-(-2))\mathbf{j}+(-1-3)\mathbf{k}$

$=-3\mathbf{i}+5\mathbf{j}-4\mathbf{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4\mathbf{i}-7\mathbf{j}+7\mathbf{k})-(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

$=(4-1)\mathbf{i}+(-7-(-2))\mathbf{j}+(7-3)\mathbf{k}$

$=3\mathbf{i}-5\mathbf{j}+4\mathbf{k}$

we see $\overrightarrow{AB}=-\overrightarrow{AC}$, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Since they contain a common point A, they must lie on the same line.

2 a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}5\\1\\5\end{pmatrix}-\begin{pmatrix}2\\3\\-3\end{pmatrix}=\begin{pmatrix}3\\-2\\8\end{pmatrix}$

b $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}8\\-1\\13\end{pmatrix}-\begin{pmatrix}2\\3\\-3\end{pmatrix}=\begin{pmatrix}6\\-4\\16\end{pmatrix}$

we see $\overrightarrow{AC}=2\overrightarrow{AB}$, so $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel. Since they contain a common point A, then A, B, & C are collinear.

3 $\overrightarrow{P_1P_2}=\overrightarrow{OP}_2-\overrightarrow{OP}_1=\begin{pmatrix}-2\\1\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-3\\-1\\0\end{pmatrix}$

$\overrightarrow{P_1P_3}=\overrightarrow{OP}_3-\overrightarrow{OP}_1=\begin{pmatrix}-5\\0\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-6\\-2\\0\end{pmatrix}$

we see $\overrightarrow{P_1P_3}=2\overrightarrow{P_1P_2}$. Since they contain a common point, they are collinear.

Since P_4 collinear with P_1, P_2, P_3, we have

$\overrightarrow{P_1P_4}=k\overrightarrow{P_1P_2}$ for some $k\in R$

$\overrightarrow{P_1P_4}=\begin{pmatrix}2\\s\\t\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}1\\s-2\\t-4\end{pmatrix}$ for some s & t

Now $\begin{pmatrix} 1 \\ s-2 \\ t-4 \end{pmatrix} = k\begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

$1 = -3k \Rightarrow k = \frac{-1}{3}$

$s - 2 = -k \Rightarrow s = 2 - k = 2 + \frac{1}{3} = \frac{7}{3}$

$t - 4 = 0 \Rightarrow t = 4$

$\therefore P_4 = \left(2, \frac{7}{3}, 4\right)$

4 $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}, \overrightarrow{OB} = x\mathbf{i}, \overrightarrow{OC} = \mathbf{i} - 2\mathbf{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x-3)\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (1-3)\mathbf{i} + (-2-4)\mathbf{j}$

$= -2\mathbf{i} - 6\mathbf{j}$

If A, B, C are collinear, $\overrightarrow{AB} = k\overrightarrow{AC}$ for some $k \in R$

$\therefore (x-3)\mathbf{i} - 4\mathbf{j} = k(-2\mathbf{i} - 6\mathbf{j})$

$\mathbf{j}$ components $\Rightarrow -4 = -6k \Rightarrow k = \frac{2}{3}$

so $x - 3 = -2k = \frac{-4}{3}$

$x = \frac{9}{3} - \frac{4}{3} = \frac{5}{3}$

so $\overrightarrow{AB} = \frac{-4}{3}\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (\mathbf{i} - 2\mathbf{j}) - \frac{5}{3}\mathbf{i} = \frac{-2}{3}\mathbf{i} - 2\mathbf{j}$

$\overrightarrow{AB} : \overrightarrow{BC} = \begin{pmatrix} \frac{-4}{3} \\ -4 \end{pmatrix} : \begin{pmatrix} \frac{-2}{3} \\ -2 \end{pmatrix}$

$= 2:1$

Exercise 12E

1 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

Distance $AB = \sqrt{5^2 + (-2)^2}$

$= \sqrt{29}$

≈ 5.39

2 $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 2 \end{pmatrix}$

Distance $AB = \sqrt{11^2 + 2^2 + 2^2} = \sqrt{129}$

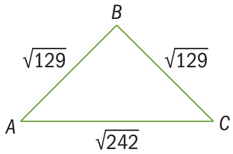
$\overrightarrow{AC} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ -3 \end{pmatrix}$

Distance $AC = \sqrt{13^2 + 8^2 + (-3)^2} = \sqrt{242}$

$\overrightarrow{BC} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -5 \end{pmatrix}$

Distance $BC = \sqrt{2^2 + 10^2 + 5^2} = \sqrt{129}$

Distance AB = Distance BC, so ABC is isosceles.



$\cos\left(C\hat{A}B\right) = \frac{129 + 242 - 129}{2\sqrt{129}\sqrt{242}}$

$C\hat{A}B = 46.8°$

3 $|\mathbf{a}| = 7$, so $\sqrt{2^2 + (-3)^2 + t^2} = 7$

$4 + 9 + t^2 = 49$

$t^2 = 36$

$t = \pm 6$

4 $\mathbf{a} = x\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$|\mathbf{a}| = \sqrt{x^2 + 6^2 + (-2)^2} = 3x$

$x^2 + 36 + 4 = 9x^2$

$8x^2 = 40$

$x^2 = \pm\sqrt{5}$

5 $|\mathbf{u}| = |\mathbf{v}|$, so

$a^2 + (-a)^2 + (2a)^2 = 2^2 + (-4)^2 + (-2)^2$

$a^2 + a^2 + 4a^2 = 4 + 16 + 4$

$6a^2 = 24$

$a^2 = 4$

$a = \pm 2$

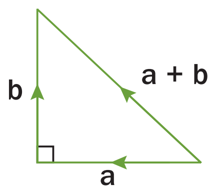
6 **a** $\mathbf{b} = 2\mathbf{a}$

Then $|\mathbf{a} + \mathbf{b}| = |3\mathbf{a}| = 3|\mathbf{a}| = 15$

b $\mathbf{b} = -3\mathbf{a}$

Then $|\mathbf{a} + \mathbf{b}| = |-2\mathbf{a}| = 2|\mathbf{a}| = 10$

c



Using Pythagoras

$|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2$

Hence $|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 12^2} = 13$

Exercise 12F

1 $\left|\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1} = 1$

2 $\left|\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right| = \sqrt{\frac{1}{3^2} + \frac{2^2}{3^2} + \frac{2^2}{3^2}}$

$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$

$= \sqrt{1} = 1$

3 $|4\mathbf{i} - 3\mathbf{j}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

So unit vector is $\frac{1}{5}(4\mathbf{i} - 3\mathbf{j}) = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

4 $\left|\begin{pmatrix}-1\\-5\\4\end{pmatrix}\right| = \sqrt{(-1)^2+(-5)^2+4^2} = \sqrt{42}$

So unit vector is $\frac{1}{\sqrt{42}}\begin{pmatrix}-1\\-5\\4\end{pmatrix}$

5 $\overline{P_1P_2} = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$

$\left|\overline{P_1P_2}\right| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$

So unit vector is $\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$

6 $|a\mathbf{i}+2a\mathbf{j}| = \sqrt{a^2+(2a)^2} = \sqrt{5a^2}$

$= \sqrt{5}a$

Now $\sqrt{5}a = 1$, so $a = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

7 $|2\mathbf{i}-\mathbf{j}| = \sqrt{2^2+(-1)^2}$

$= \sqrt{5}$

So unit vector is $\frac{1}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})$

Vector of magnitude 5 is $\frac{5}{\sqrt{5}}(2\mathbf{i}-\mathbf{j}) = \sqrt{5}(2\mathbf{i}-\mathbf{j})$

8 $\left|\begin{pmatrix}-1\\-3\\2\end{pmatrix}\right| = \sqrt{(-1)^2(-3)^2+2^2} = \sqrt{14}$

unit vector is $\frac{1}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

and vector magnitude 7 is $\frac{7}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix} = \frac{\sqrt{14}}{2}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

9 **a** $\left|\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}\right| = \sqrt{2^2\cos^2\theta + 2^2\sin^2\theta}$

$= \sqrt{4(\cos^2\theta+\sin^2\theta)}$

$= 2\sqrt{(1)} = 2$

So unit vector is $\frac{1}{2}\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}$ or $\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

b $\left|\begin{pmatrix}1\\\tan\alpha\end{pmatrix}\right| = \sqrt{1^2+\tan^2\alpha}$

$= \sqrt{\sec^2\alpha}$

$= \sec\alpha = \frac{1}{\cos\alpha}$

So unit vector is $\frac{1}{\sec\alpha}\begin{pmatrix}1\\\tan\alpha\end{pmatrix} = \begin{pmatrix}\cos\alpha\\\sin\alpha\end{pmatrix}$

Exercise 12G

1 **a** $\mathbf{a}+\mathbf{b} = (2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})$

$= (2+3)\mathbf{i}+(-1+2)\mathbf{j}$

$= 5\mathbf{i}+\mathbf{j}$

b $\mathbf{b}+\mathbf{c} = (3\mathbf{i}+2\mathbf{j})+(-\mathbf{i}+\mathbf{j})$

$= (3-1)\mathbf{i}+(2+1)\mathbf{j}$

$= 2\mathbf{i}+3\mathbf{j}$

c $\mathbf{c}+\mathbf{d} = (-\mathbf{i}+\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$= (-1+3)\mathbf{i}+(1+3)\mathbf{j}$

$= 2\mathbf{i}+4\mathbf{j}$

d $\mathbf{a}+\mathbf{b}+\mathbf{d} = (2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$= (2+3+3)\mathbf{i}+(-1+2+3)\mathbf{j}$

$= 8\mathbf{i}+4\mathbf{j}$

e $\mathbf{a}-\mathbf{b} = (2\mathbf{i}-\mathbf{j})-(3\mathbf{i}+2\mathbf{j})$

$= (2-3)\mathbf{i}+(-1-2)\mathbf{j}$

$= -\mathbf{i}-3\mathbf{j}$

f $\mathbf{d}-\mathbf{b}+\mathbf{a} = (3\mathbf{i}+3\mathbf{j})-(3\mathbf{i}+2\mathbf{j})+(2\mathbf{i}-\mathbf{j})$

$= (3-3+2)\mathbf{i}+(3-2-1)\mathbf{j}$

$= 2\mathbf{i}+0\mathbf{j} = 2\mathbf{i}$

2 **a** $\mathbf{a}+\mathbf{b} = \begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-4\\5\end{pmatrix} = \begin{pmatrix}2 & -4\\-3 & +5\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$

b $\mathbf{b}-\mathbf{c} = \begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix} = \begin{pmatrix}-4 & -(-5)\\5 & -(-3)\end{pmatrix} = \begin{pmatrix}1\\8\end{pmatrix}$

c $\frac{1}{2}(\mathbf{a}+\mathbf{c}) = \frac{1}{2}\left[\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-5\\-3\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}2 & -5\\-3 & +(-3)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-3\\-6\end{pmatrix}$

d $\mathbf{a}+3\mathbf{b}-\mathbf{c} = \begin{pmatrix}2\\-3\end{pmatrix}+3\begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}$

$= \begin{pmatrix}2+3(-4)-(-5)\\-3+3(5)-(-3)\end{pmatrix}$

$= \begin{pmatrix}-5\\15\end{pmatrix}$

e $3\mathbf{c} - 2\mathbf{b} + 5\mathbf{a} = 3\begin{pmatrix} -5 \\ -3 \end{pmatrix} - 2\begin{pmatrix} -4 \\ 5 \end{pmatrix} + 5\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 3(-5) -2(-4) \ \ +5(2) \\ 3(-3) \ \ -2(5) +5(-3) \end{pmatrix}$

$= \begin{pmatrix} -15 \ \ +8 +10 \\ -9 -10 -15 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -34 \end{pmatrix}$

3 **a** $\mathbf{a} + \mathbf{b} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + (5\mathbf{i} - \mathbf{k})$

$= (3 + 5)\mathbf{i} + (-1)\mathbf{j} + (-2-1)\mathbf{k}$

$= 8\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

b $\mathbf{b} - 2\mathbf{a} = (5\mathbf{i} - \mathbf{k}) - 2(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= (5 - 6)\mathbf{i} - 2(-1)\mathbf{j} + (-1-2(-2))\mathbf{k}$

$= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

c $2\mathbf{a} - \mathbf{b} = 2(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) - (5\mathbf{i} - \mathbf{k})$

$= (6 - 5)\mathbf{i} + (-2)\mathbf{j} + (-4 + 1)\mathbf{k}$

$= \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

d $4(\mathbf{a} - \mathbf{b}) + 2(\mathbf{b} + \mathbf{a}) = 4((3 - 5)\mathbf{i} - \mathbf{j} + (-2 + 1)\mathbf{k})$

$+ 2(8\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ from Q3a

$= -8\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} + 16\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

$= (-8 + 16)\mathbf{i} - (4 + 2)\mathbf{j} + (-4 - 6)\mathbf{k}$

$= 8\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}$

4 **a** $2\mathbf{x} - 3\mathbf{p} = \mathbf{q}$

$2\mathbf{x} - 3\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$2\mathbf{x} = \begin{pmatrix} -1 \ +9 \\ 4 -15 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \end{pmatrix}$

$\mathbf{x} = \begin{pmatrix} 4 \\ -5.5 \end{pmatrix}$

b $4\begin{pmatrix} 3 \\ -5 \end{pmatrix} - 3(\mathbf{y}) = 7\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 12 \\ -20 \end{pmatrix} - 3\mathbf{y} = \begin{pmatrix} -7 \\ 28 \end{pmatrix}$

$\begin{pmatrix} 12 \\ -20 \end{pmatrix} - \begin{pmatrix} -7 \\ 28 \end{pmatrix} = 3\mathbf{y}$

$\begin{pmatrix} 19 \\ -48 \end{pmatrix} = 3\mathbf{y}$

So $\mathbf{y} = \frac{1}{3}(19\mathbf{i} - 48\mathbf{j})$

c $2\mathbf{p} + \mathbf{z} = 0$

$2\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$6 + \mathbf{z}_1 = 0 \Rightarrow \mathbf{z}_1 = -6$

$-10 + \mathbf{z}_2 = 0 \Rightarrow \mathbf{z}_2 = 10$

$\mathbf{z} = -6\mathbf{i} + 10\mathbf{j}$

5 $\mathbf{a} = \mathbf{b} \Rightarrow \begin{pmatrix} x \\ x + y \end{pmatrix} = \begin{pmatrix} 6 - y \\ -2x - 3 \end{pmatrix}$

$x = 6 - y$ (1)

$x + y = -2x - 3$

$y = -3x - 3$ (2)

Sub (1) into (2)

$y = -3(6 - y) - 3$

$y = -18 + 3y - 3$

$-2y = -21$

$y = \frac{21}{2}$

$x = 6 - y = 6 - \left(\frac{21}{2}\right) = \frac{-9}{2}$

6 $3\mathbf{a} = 2\mathbf{b} \Rightarrow 3\begin{pmatrix} 3 \\ t \\ u \end{pmatrix} = 2\begin{pmatrix} t - s \\ 3s \\ t + s \end{pmatrix}$

(1) $9 = 2(t - s)$

(2) $3t = 6s$

(3) $3u = 2(t + s)$

(2) $\Rightarrow t = 2s$

(1) $\Rightarrow 9 = 2(2s - s)$

$= 2s$

$s = \frac{9}{2}$

(2) $\Rightarrow t = 9$

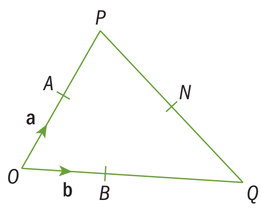
(3) $\Rightarrow 3u = 2\left(9 + \frac{9}{2}\right)$

$u = \frac{27}{3} = 9$

$t = 9, s = \frac{9}{2}, u = 9$

Exercise 12H

1



a $\overrightarrow{AP} = \overrightarrow{OA} = \mathbf{a}$

b $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

c $\overrightarrow{PQ} = -\overrightarrow{AP} - \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BQ}$
$= -\mathbf{a} - \mathbf{a} + \mathbf{b} + 3\mathbf{b}$
$= 4\mathbf{b} - 2\mathbf{a}$

d $\overrightarrow{PN} = \frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(4\mathbf{b} - 2\mathbf{a})$
$= 2\mathbf{b} - \mathbf{a}$

e $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PN}$
$= \mathbf{a} + \mathbf{a} + (2\mathbf{b} - \mathbf{a})$
$= \mathbf{a} + 2\mathbf{b}$

f $\overrightarrow{AN} = \overrightarrow{AP} + \overrightarrow{PN}$
$= \mathbf{a} + (2\mathbf{b} - \mathbf{a})$
$= 2\mathbf{b}$

2 $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \ \overrightarrow{AC} : \overrightarrow{CB} = 3:1$

a $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

b $\overrightarrow{AC} = \frac{3}{4}\overrightarrow{AB}$
$= \frac{3}{4}(\mathbf{b} - \mathbf{a})$

c $\overrightarrow{CB} = \frac{1}{4}\overrightarrow{AB}$
$= \frac{1}{4}(\mathbf{b} - \mathbf{a})$

d $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$
$= \mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a})$
$= \mathbf{a}\left(1 - \frac{3}{4}\right) + \mathbf{b}\frac{3}{4}$
$= \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

3 $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OC} = \mathbf{c}, \overrightarrow{CB} = 3\mathbf{a}$

a $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$
$= \mathbf{c} + 3\mathbf{a}$

b $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{CB}$
$= -\mathbf{a} + \mathbf{c} + 3\mathbf{a}$
$= \mathbf{c} + 2\mathbf{a}$

c $\overrightarrow{OD} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
$= \mathbf{a} + \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} + \frac{1}{2}\mathbf{c}$

d $\overrightarrow{CD} = \overrightarrow{CB} - \frac{1}{2}\overrightarrow{AB}$
$= 3\mathbf{a} - \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} - \frac{1}{2}\mathbf{c}$

4 **a** $\overrightarrow{FA} = \mathbf{a}, \overrightarrow{FB} = \mathbf{b}$

i $\overrightarrow{AB} = -\overrightarrow{FA} + \overrightarrow{FB}$
$= -\mathbf{a} + \mathbf{b}$

ii $\overrightarrow{FO} = \overrightarrow{FB} + \overrightarrow{BO}$
By symmetry, $\overrightarrow{OB} = \overrightarrow{FA} = \mathbf{a}$
so $\overrightarrow{FO} = \overrightarrow{FB} - \overrightarrow{OB}$
$= \mathbf{b} - \mathbf{a}$

iii $\overrightarrow{FC} = \overrightarrow{FO} + \overrightarrow{OE} + \overrightarrow{EC}$
By symmetry, $\overrightarrow{OE} = \overrightarrow{BO} = -\mathbf{a}$
and $\overrightarrow{EC} = \overrightarrow{FB} = \mathbf{b}$
so $\overrightarrow{FC} = (\mathbf{b} - \mathbf{a}) - \mathbf{a} + \mathbf{b}$
$= 2(\mathbf{b} - \mathbf{a})$

iv $\overrightarrow{BC} = \overrightarrow{BE} + \overrightarrow{EC}$
$= \overrightarrow{BO} + \overrightarrow{OC} + \overrightarrow{EC}$
$= -\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{EC}$
$= -\mathbf{a} - \mathbf{a} + \mathbf{b}$
$= \mathbf{b} - 2\mathbf{a}$

v $\overrightarrow{FD} = \overrightarrow{FC} + \overrightarrow{CD}$
By symmetry, $\overrightarrow{CD} = -\overrightarrow{FA} = -\mathbf{a}$
so $\overrightarrow{FD} = 2(\mathbf{b} - \mathbf{a}) - \mathbf{a} = 2\mathbf{b} - 3\mathbf{a}$

b AB is parallel to and half the length of FC

c $\overrightarrow{FD} = 2\mathbf{b} - 3\mathbf{a}$
$\overrightarrow{AC} = \overrightarrow{AF} + \overrightarrow{FC}$ (see **iii**)
$= -\mathbf{a} + 2(\mathbf{b} - \mathbf{a})$
$= -3\mathbf{a} + 2\mathbf{b}$
$\overrightarrow{FD} = \overrightarrow{AC}$ ∴ FD and AC are parallel

5 $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

a $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$
since $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$
$\overrightarrow{AP} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$

b M is mid point of OA, so
$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$
$\overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{AP}$
$= \frac{1}{2}\mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$
$= \left(\frac{1}{2} - \frac{2}{3}\right)\mathbf{a} + \frac{2}{3}\mathbf{b}$
$= \frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

c $\overrightarrow{MX} = \overrightarrow{MP} + \overrightarrow{PB} + \overrightarrow{BX}$
$\overrightarrow{PB} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$
$\overrightarrow{BX} = \overrightarrow{OB} = \mathbf{b}$
so $\overrightarrow{MX} = \left(\frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}\right) + \frac{1}{3}(\mathbf{b} - \mathbf{a}) + \mathbf{b}$
$= 2\mathbf{b} - \frac{1}{2}\mathbf{a}$

d $\overrightarrow{MP} = \frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

$\overrightarrow{MX} = 2\mathbf{b} - \frac{1}{2}\mathbf{a}$

$\overrightarrow{MX} = 3\overrightarrow{MP}$ $\therefore$ MX is parallel to MP

Since MX and MP share the common point M, MPX is a straight line

Exercise 12I

1 a $\mathbf{a}\cdot\mathbf{b} = (2\mathbf{i}+4\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})$
$= (2\times 1) + (4\times -5)$
$= 2 - 20$
$= -18$

b $\mathbf{b}\cdot\mathbf{c} = (\mathbf{i}-5\mathbf{j})\cdot(-5\mathbf{i}-2\mathbf{j})$
$= (1\times -5) + (-5\times -2)$
$= -5 + 10$
$= 5$

c $\mathbf{a}\cdot\mathbf{a} = (2\mathbf{i}+4\mathbf{j})\cdot(2\mathbf{i}+4\mathbf{j})$
$= (2\times 2) + (4\times 4)$
$= 4 + 16$
$= 20$

d $\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = (-5\mathbf{i}-2\mathbf{j})\cdot[(2\mathbf{i}+4\mathbf{j}) + (\mathbf{i}-5\mathbf{j})]$
$= (-5\mathbf{i}-2\mathbf{j})\cdot[3\mathbf{i}-\mathbf{j}]$
$= (-5\times 3) + (-2\times -1)$
$= 15 + 2$
$= -13$

e $(\mathbf{c}+\mathbf{a})\cdot\mathbf{b} = [(-5\mathbf{i}-2\mathbf{j}) + (2\mathbf{i}+4\mathbf{j})]\cdot(\mathbf{i}-5\mathbf{j})$
$= [-3\mathbf{i}+2\mathbf{j}]\cdot(\mathbf{i}-5\mathbf{j})$
$= (-3\times 1) + (2\times -5)$
$= -3 - 10 = -13$

2 a $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4\\-3\\-1\end{pmatrix} = (-1\times 4) + (0\times -3) + (5\times -1)$
$= -4 + 0 - 5$
$= -9$

b $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4 & -(-1)\\-3 & -(3)\\-1 & -(-6)\end{pmatrix} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\5\end{pmatrix}$
$= (-1\times 5) + (0\times -6) + (5\times 5)$
$= -5 + 0 + 25$
$= 20$

c $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = -9 - \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-6\end{pmatrix}$
$= -9 - [(-1)\times(-1) + 0\times 3 + 5\times(-6)]$
$= -9 - [1 + 0 - 30]$
$= -9 + 29 = 20$

d $2\,\mathbf{u}\cdot\mathbf{w} = 2(-29)$
$= -58$

e $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{w}) = \begin{bmatrix}-1 & -4\\0 & -(-3)\\5 & -(-1)\end{bmatrix}\cdot\begin{bmatrix}-1 & +(-1)\\0 & +3\\5 & +(-6)\end{bmatrix}$
$= \begin{pmatrix}-5\\3\\6\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\-1\end{pmatrix}$
$= 10 + 9 - 6 = 13$

3 a $\mathbf{a}\cdot\mathbf{b} = (2\times 4) + (4\times -2)$
$= 8 - 8$
$= 0 \Rightarrow$ perpendicular.

b $\mathbf{c}\cdot\mathbf{d} = \begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = (2\times 1) + (1\times 2) = 4$

$|\mathbf{c}| = \sqrt{2^2+1^2} = \sqrt{5}$ $|\mathbf{d}| = \sqrt{5}$

$|\mathbf{c}|\,|\mathbf{d}| = \left(\sqrt{5}\right)^2 = 5 \neq \mathbf{c}\cdot\mathbf{d}$

So neither parallel, nor perpendicular.

c $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-8\\2\\2\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\-1\end{pmatrix} = (-8\times 4) + (2\times -1) + (2\times -1)$
$= -32 - 2 - 2 = -36$

$|\mathbf{u}| = \sqrt{8^2+2^2+2^2} = \sqrt{64+4+4} = \sqrt{72}$

$|\mathbf{v}| = \sqrt{4^2+1^2+1^2} = \sqrt{16+1+1} = \sqrt{18}$

$|\mathbf{u}||\mathbf{v}| = \sqrt{18\times 72} = 36 = -\mathbf{u}\cdot\mathbf{v}$

$\Rightarrow$ parallel.

d $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$\mathbf{a}\cdot\mathbf{b} = (3\times 3) + (-2\times -2) + (1\times -1)$
$= 9 + 4 - 1$
$= 12$

$|\mathbf{a}| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$

$|\mathbf{a}||\mathbf{b}| = \left(\sqrt{14}\right)^2 = 14 \neq \mathbf{a}\cdot\mathbf{b}$

$\Rightarrow$ neither parallel, nor perpendicular

e $\overrightarrow{OX}\cdot\overrightarrow{OZ} = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = (1\times 0) + (0\times 0) + (0\times 1) = 0$

$\Rightarrow$ perpendicular

f $\mathbf{n}\cdot\mathbf{m} = (2\mathbf{i} - 8\mathbf{j})(-\mathbf{i} + 4\mathbf{j})$
$= (2\times -1) + (-8\times 4) = -2 - 32 = -34$

$|\mathbf{n}| = \sqrt{2^2+8^2} = \sqrt{4+64} = \sqrt{68}$

$|\mathbf{m}| = \sqrt{1^2+4^2} = \sqrt{1+16} = \sqrt{17}$

$|\mathbf{n}||\mathbf{m}| = \sqrt{17 \times 68} = 34 = -\mathbf{n} \cdot \mathbf{m}$

$\Rightarrow$ parallel.

g $\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} = (2 \times -1) + (2 \times -1)$

$= -2 - 2$

$= -4$

$|\overrightarrow{AB}| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$|\overrightarrow{CD}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|\overrightarrow{AB}| \cdot |\overrightarrow{CD}| = \sqrt{2 \times 8} = \sqrt{16} = 4 = -\overrightarrow{AB} \cdot \overrightarrow{CD}$

$\Rightarrow$ parallel vectors

4 $\mathbf{a} + 3\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + 3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= (1 + 3 \times 3)\mathbf{i} + (1 + 3 \times 2)\mathbf{j} + (2 + 3 \times -1)\mathbf{k}$

$= 10\mathbf{i} + 7\mathbf{j} - \mathbf{k}$

$2\mathbf{a} - \mathbf{b} = 2(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= (2(1) - 3)\mathbf{i} + (2(1) - 2)\mathbf{j} + (2(2) - (-1))\mathbf{k}$

$= -\mathbf{i} + 5\mathbf{k}$

$(\mathbf{a} + 3\mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b}) = (10\mathbf{i} + 7\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{k})$

$= (10 \times -1) + (-1 \times 5)$

$= -10 - 5 = -15$

5 Let $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

$\mathbf{a} \cdot \mathbf{d} = 3d_1 + (-5)d_3 = -9$

$\mathbf{b} \cdot \mathbf{d} = 2d_1 + 7d_2 = 11$

$\mathbf{c} \cdot \mathbf{d} = d_1 + d_2 + d_3 = 6$

using GDC, $d_1 = 2$, $d_2 = 1$, $d_3 = 3$

So, $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

6 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta$

$\sqrt{6} = 2\sqrt{3}\cos\theta$

$\cos\theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$

7 **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2 \times 2 + -1 \times 5 = 4 - 5 = -1$

$\left|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\left|\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$

$-1 = \sqrt{5}\sqrt{29}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{145}}\right) = 94.8°$

b $\begin{pmatrix} 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} = 4 \times -3 + 0 \times 1 = -12$

$\left|\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right| = \sqrt{4^2} = 4$

$\left|\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$-12 = 4\sqrt{10}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) = 161.6°$

c $(2\mathbf{i} + 5\mathbf{j}) \cdot (2\mathbf{i} - 5\mathbf{j}) = (2 \times 2) + (5 \times -5)$

$= 4 - 25$

$= -21$

$|(2\mathbf{i} + 5\mathbf{j})| = \sqrt{2^2 + 5^2} = \sqrt{29} = |(2\mathbf{i} - 5\mathbf{j})|$

$\therefore -21 = 29\cos\theta$

$\theta = \cos^{-1}\left(\frac{-21}{29}\right) = 136.4°$

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

b $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = (-1 \times 1) + (5 \times -2)$

$= -1 - 10 = -11$

c $|\overrightarrow{AB}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$|\overrightarrow{AC}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\cos\theta$

$-11 = \sqrt{5 \times 26}\cos\theta$

$\cos\theta = \frac{-11}{\sqrt{130}}$

9 **a** $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -2 - 6 + 12 = 4$

$\left|\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\left|\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = 7$

so $4 = 3 \times 7\cos\theta$

$\cos\theta = \frac{4}{21}, \theta = 79°$

b $\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\-2\end{pmatrix}=8-6-2=0 \Rightarrow$ perpendicular vectors

$\theta = 90°$

c $(2\mathbf{i}-7\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})$

$=(2\times1)+(-7\times1)+(1\times-1)=-6$

$|(2\mathbf{i}-7\mathbf{j}+\mathbf{k})|=\sqrt{2^2+7^2+1^2}=\sqrt{54}$

$|(\mathbf{i}+\mathbf{j}-\mathbf{k})|=\sqrt{1+1+1}=\sqrt{3}$

so $-6=\sqrt{162}\cos\theta$

$\theta=\cos^{-1}\left(\frac{-6}{\sqrt{162}}\right)=118.1°$

10 a $\overrightarrow{AB}=\begin{pmatrix}2\\3\\4\end{pmatrix}-\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\4\\0\end{pmatrix}$

$\overrightarrow{AB}==\sqrt{1^2+4^2}=\sqrt{17}$

$\overrightarrow{AC}=\begin{pmatrix}2\\-1\\-1\end{pmatrix}-\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\0\\-5\end{pmatrix}$

$\overrightarrow{AC}=\sqrt{1^2+(-5)^2}=\sqrt{26}$

so $AB=\sqrt{17}$, $AC=\sqrt{26}$

b $\overrightarrow{AB}\cdot\overrightarrow{AC}=\begin{pmatrix}1\\4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-5\end{pmatrix}=1$

$\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$

$1=\sqrt{17}\sqrt{26}\cos\theta$

$\frac{1}{\sqrt{442}}=\cos\theta$

c Area $ABC=\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin B\hat{A}C$

$=\frac{1}{2}\sqrt{442}\sin\left(\cos^{-1}\frac{1}{\sqrt{442}}\right)=10.5\text{cm}^2$

11 The x-axis has unit direction vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$

so $\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=1$

$\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|=\sqrt{1^2}=1$ and $\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|=\sqrt{1+1+1}=\sqrt{3}$

so $1=\sqrt{3}\cos\theta$

$\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)=54.7°$

12 a $\overrightarrow{OA}=4\mathbf{i}+4\mathbf{j}-4\mathbf{k}$, $\overrightarrow{OB}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

$(\overrightarrow{OA}).(\overrightarrow{OB})=(4\times1)+(4\times2)+(-4\times3)$

$=4+8-12=0$

$=\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular

b $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=(1-4)\mathbf{i}+(2-4)\mathbf{j}+(3-(-4))\mathbf{k}$

$=-3\mathbf{i}-2\mathbf{j}+7\mathbf{k}$

$|\overrightarrow{AB}|=\sqrt{(-3)^2+(-2)^2+7^2}=\sqrt{62}\approx7.87$

13 $(2\mathbf{i}+\lambda\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}+3\mathbf{k})=(2\times1)+(\lambda\times-2)+(3\times1)$

$=2-2\lambda+3=0$ for perpendicular vectors.

$5=2\lambda$

$\lambda=\frac{5}{2}$

14 a $\mathbf{a}+\mathbf{b}=(5+1)\mathbf{i}+(-3+1)\mathbf{j}+(7+\lambda)\mathbf{k}$

$=6\mathbf{i}-2\mathbf{j}+(7+\lambda)\mathbf{k}$

$\mathbf{a}-\mathbf{b}=(5-1)\mathbf{i}+(-3-1)\mathbf{j}+(7-\lambda)\mathbf{k}$

$=4\mathbf{i}-4\mathbf{j}+(7-\lambda)\mathbf{k}$

Now $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=(6\times4)+(-2\times-4)+(\lambda+7)(7-\lambda)$

$=24+8+49-\lambda^2=0$

$\lambda^2=81$

$\lambda=\pm9$

15 $\mathbf{a}+\mathbf{b}=\begin{pmatrix}p\\2\\-p\end{pmatrix}+\begin{pmatrix}2\\-p\\-3\end{pmatrix}=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}$

$\mathbf{a}-\mathbf{b}=\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}\cdot\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$=(p^2-4)+(4-p^2)-(9-p^2)$

$=p^2-9=0$ for perpendicular vectors.

$\Rightarrow p^2=9,\ p=\pm3$

Exercise 12J

1 a $\mathbf{r}=\begin{pmatrix}-1\\2\end{pmatrix}+t\begin{pmatrix}3\\2\end{pmatrix}, t\in\mathbb{R}.$

b $\mathbf{r}=\begin{pmatrix}-1\\0\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix}, t\in\mathbb{R}.$

c $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}, t \in \mathbb{R}.$

d $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k}), t \in \mathbb{R}.$

2 **a** Position vectors are $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Line joining the 2 points has direction

$\begin{pmatrix} 3 & -4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 7 \end{pmatrix}, t \in \mathbb{R}.$

b Position vectors $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 5 & -4 \\ -2 & (+2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$

c Position vectors $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 3 & -2 \\ 5 & -(-4) \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix}, t \in \mathbb{R}.$

d Position vectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 0 & -1 \\ 0 & -(-1) \\ 1 & -0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

3 **a** We need a vector $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ which is perpendicular to $\mathbf{a}$

$\mathbf{a} \cdot \mathbf{p} = 0 \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 3p_1 + 2p_2 = 0$

Take $p_1 = 2, p_2 = -3$

Then $\mathbf{a} \cdot \mathbf{p} = 0$, and line is

$\mathbf{r} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \end{pmatrix}, t \in \mathbb{R}.$

b Using the same technique as in part **a**, we see $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is perpendicular $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}, t \in \mathbb{R}.$

c $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

line is $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, t \in \mathbb{R}.$

d We require $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ so that $\mathbf{p} \cdot \mathbf{a} = 0$

$\mathbf{p} \cdot \mathbf{a} = p_1 - 3p_2 + 4p_3 = 0$

Take $p_1 = 0, p_2 = 4, p_3 = 3$ for example

Then line is $\mathbf{r} = 5\mathbf{k} + t(4\mathbf{j} + 3\mathbf{k}), t \in \mathbb{R}.$

4 **a** We need to know if there is a value of t for which

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Take $t = 2$ Then $\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

so (4, 5) lies on the line.

b Is there t so that $\begin{pmatrix} 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$?

$5 + t(4) = 5$ and $1 - 3t = -2$

$t = 0$ and $t = 1 \Rightarrow$ no such t.

so (5, –2) does not lie on the line.

c Is there t so that $\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$?

$-1 + t = -3 \Rightarrow t = -2$

$5 + 0(t) = 5 \Rightarrow t =$ anything

$-3 - 2t = 1 \Rightarrow t = -2$

so $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ i.e. (–3, 5, 1) lies on line.

d Is there t so that

$(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + t(-2\mathbf{j} - 3\mathbf{k})$

$1 = -1 - 2t$ and $1 = -3 - 3t$

$-2 = 2t \qquad 4 = -3t$

$t = -1$ and $t = \frac{-4}{3} \Rightarrow$ no such t.

so (2,1,1) does not lie on line.

5 $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}, t \in \mathbb{R}$

$10 = 4 + 3t \Rightarrow 6 = 3t, t = 2$

$p = 2 - 2t = 2 - 2(2) = -2$

$q = 5 + 8t = 5 + 8(2) = 21$

6 A vertical line will have direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\mathbf{r} = \begin{pmatrix} -6 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}, t \in \mathbb{R}$

7 a (1) Are the 2 lines parallel?

Is there t such that $\begin{pmatrix} 2 \\ -1 \end{pmatrix} = t\begin{pmatrix} -6 \\ 3 \end{pmatrix}$

Take $t = \frac{-1}{3}$. Then $\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ so lines parallel.

(2) Are 2 lines co-incident?

Does $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix} -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Take $s = -6$.

$\begin{pmatrix} -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 6\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

so $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ lies on $\mathbf{r}_1 \Rightarrow$ lines co-incident

b (1) Are lines parallel?

Is there t so that $\begin{pmatrix} -4 \\ 2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow$ No such t, so NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -4 + 4 = 0 \Rightarrow$ perpendicular.

c (1) Are the lines parallel?

Is there t so that $\begin{pmatrix} 4 \\ -3 \end{pmatrix} = t\begin{pmatrix} 8 \\ -6 \end{pmatrix}$

$t = 2$ gives

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 8 \\ -6 \end{pmatrix} \Rightarrow$ lines parallel.

(2) Are lines co-incident?

Does $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + s\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ No such $s \Rightarrow$

lines NOT co-incident.

d (1) Are lines parallel?

Is there t so that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ No such t

$\Rightarrow$ NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + 2 = 3 \Rightarrow$ NOT perpendicular.

e (1) Are lines parallel?

Is there t so that

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} = t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ No such $t \Rightarrow$

lines not parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 16 - 9 = 7 \Rightarrow$

NOT perpendicular.

8 a $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right| \cos A$

$2 + 4 = \sqrt{1^2 + 4^2}\sqrt{2^2 + 1^2 + 1^2} \cos A$

$6 = \sqrt{17}\sqrt{6} \cos A$

$A = \cos^{-1}\left(\frac{6}{\sqrt{17}\sqrt{6}}\right) = 53.6°$

b $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right| \cos A$

$-2 - 2 = \sqrt{2^2 + (-2)^2}\sqrt{(-1)^2 + 3^2 + 1^2} \cos A$

$-4 = \sqrt{8}\sqrt{11} \cos A$

$A = \cos^{-1}\left(\frac{-4}{\sqrt{8}\sqrt{11}}\right) = 115.2°$

9 a A has position vector $\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}$. We require t so that

$\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ Taking $t = -1$, we see:

$-2 = -1 + (-1)1$
$-3 = -1 + (-1)2$
$-4 = 2 + (-1)\,6$

b $\overrightarrow{AB} = \begin{pmatrix} -6 \\ -7 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 2 \end{pmatrix}$

Taking dot product,

$\begin{pmatrix} -4 \\ -4 \\ 2 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = -4 - 8 + 12 = 0$

$\Rightarrow \overrightarrow{AB}$ perpendicular toL_1

10 a i $\overrightarrow{OF} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

ii $\overrightarrow{AG} = -2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

b i $|\overrightarrow{OF}| = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$

ii $|\overrightarrow{AG}| = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$

iii $\overrightarrow{OF} \,.\, \overrightarrow{AG} = (2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) \,.\, (-2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$
$= 30$

c $\overrightarrow{OF} \,.\, \overrightarrow{AG} = |\overrightarrow{OF}|\,|\overrightarrow{AG}|\cos\theta$

$30 = \sqrt{38}\sqrt{38}\,\cos\theta$

$\theta = \cos\theta^{-1}\left(\frac{30}{38}\right) = 7.9^\circ$

11 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 7\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$

b $\cos OAB = \dfrac{\overrightarrow{AO}.\overrightarrow{AB}}{|AO\|AB|}$
$= \dfrac{-1 \times 7 + (-5) \times (-8) + 2 \times 8}{\sqrt{30}\sqrt{177}}$
$= \dfrac{49}{\sqrt{30}\sqrt{177}}$

c Let $\mathbf{r}$ be the position vector for the point P.
Then $\mathbf{r} = (1 + 7\mu)\mathbf{i} + (5 - 8\mu)\mathbf{j} + (-2 + 8\mu)\mathbf{k}$
$= (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} + \mu(7\mathbf{i} - 8\mathbf{j} - 8\mathbf{k})$
$= \overrightarrow{OA} + \mu\overrightarrow{AB}$

which is the position vector of a point on the line that passes through A with direction vector $\overrightarrow{AB}$, and hence also passes through B.

d $\overrightarrow{OP}.\overrightarrow{AB} = 0$

$\therefore \begin{pmatrix} 1 + 7\mu \\ 5 - 8\mu \\ -2 + 8\mu \end{pmatrix} . \begin{pmatrix} 7 \\ -8 \\ 8 \end{pmatrix} = 0$

$7 + 49\mu - 40 + 64\mu - 16 + 64\mu = 0$

$\mu = \dfrac{49}{177}$

e Use the value of μ from part **d** to get:

$P = \dfrac{1}{177}\begin{pmatrix} 520 \\ 493 \\ 38 \end{pmatrix}$

Exercise 12K

1 Equating components of $\mathbf{r}_1$ & $\mathbf{r}_2$:

$4 + 2\lambda = 11 + \mu$ (1)
$2 - 4\lambda = 16 + 2\mu$ (2)
$(1) \Rightarrow \mu = -7 + 2\lambda$
$(2) \Rightarrow 2 - 4\lambda = 16 + 2(-7 + 2\lambda)$
$2 - 4\lambda = 2 + 4\lambda$
$\lambda = 0$
$(1) \Rightarrow \mu = -7$
so intercept at (4,2)

2 Equating components:

$4 + 8s = 6 + 9t$ (1)
$-2 + 2s = -3 + 6t$ (2)
$(1) \Rightarrow 8s = 2 + 9t$
$s = \frac{1}{8}(2 + 9t)$
$(2) \Rightarrow -2 + \frac{1}{4}(2 + 9t) = -3 + 6t$
$-8 + 2 + 9t = -12 + 24t$
$6 = 15t$
$t = \frac{6}{15}$
$(1) \Rightarrow s = \frac{1}{8}\left(2 + \frac{54}{15}\right) = \frac{7}{10}$

intersec at $\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \frac{7}{10}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{48}{5} \\ \frac{-3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 48 \\ -3 \end{pmatrix}$

3 $5 + 2t = 3 + 2s$ (1)
$-1 + t = -2 + s$ (2)
$2 - t = -4 + 2s$ (3)
$(2) \Rightarrow s = 1 + t$
$(1) \Rightarrow 5 + 2t = 3 + 2(1 + t)$
$5 + 2t = 5 + 2t$ (so (1) & (2) are consistent)
$(3)\ 2 - t = -4 + 2(1 + t)$
$2 - t = -2 + 2t$
$4 = 3t$
$t = \frac{4}{3}$
$(2) \Rightarrow s = 1 + \frac{4}{3} = \frac{7}{3}$
Thus l_1 & l_2 intersect.

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \frac{7}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 23 \\ 1 \\ 2 \end{pmatrix} \text{ at } \left(\frac{23}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

4 $1 + 3t = -1 \qquad (1)$

$1 - t = s \qquad (2)$

$(1) \Rightarrow 3t = -2$

$t = \frac{-2}{3}$

$(2) \Rightarrow s = 1 - \left(\frac{-2}{3}\right) = \frac{5}{3}$

Intersect at $\mathbf{i} + \frac{5}{3}\mathbf{j}$, i.e. at $\left(-1, \frac{5}{3}\right)$

5 If lines intersect then there are s & t so that

$3 - t = 1 + s \qquad (1)$

$t = 4 + s \qquad (2)$

$5 + 2t = s \qquad (3)$

sub (2) into (3): $s = 5 + 2(4 + s)$, $s = 13 + 2s$

$s = -13$

in (2) $\Rightarrow t = 4 - 13 = -9$

check in (1): $3 - (-9) \neq 1 - 13$

$12 \neq -12$

so there are no such s & $t \Rightarrow$ skew

6 **a** $3 - s = 14 + 3t \qquad (1)$

$-2 + 3s = -20 - 4t \qquad (2)$

$5 - 5s = 6 - 3t \qquad (3)$

$(1) \Rightarrow s = -11 - 3t$

$(2) \Rightarrow -2 - 3(11 + 3t) = -20 - 4t$

$-35 - 9t = -20 - 4t$

$-15 = 5t$

$t = -3$

$(1) \Rightarrow s = -11 - 3(-3) = -2$

check in (3): $5 - 5(-2) = 6 - 3(-3)$

$15 = 15$ so lines intersect.

Point of intersection $= 14\mathbf{i} - 20\mathbf{j} + 6\mathbf{k}$

$- 3(3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

when $t = -3$

$= 5\mathbf{i} - 8\mathbf{j} + 15\mathbf{k}$

b Take dot product of direction vectors:

$(-\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$= (-1 \times 3) + (3 \times -4) + (-5 \times -3)$

$= -3 - 12 + 15$

$= 0$

$\Rightarrow$ perpendicular.

7 **a** $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ a \end{pmatrix}$

$6 + t = 5 \Rightarrow t = -1$

$3 - 2(-1) = a \Rightarrow a = 5.$

$\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} b \\ 13 \\ -1 \end{pmatrix}$

$9 + 2t = 13 \Rightarrow t = 2.$

$6 + 2 = b \Rightarrow b = 8.$

b OP has position vector $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ for some t

$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 13 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$

$(\overrightarrow{OP}) \cdot (\overrightarrow{AB}) = 0 \Rightarrow \begin{pmatrix} 6+t \\ 9+2t \\ 3-2t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = 0$

$3(6 + t) + 6(9 + 2t) - 6(3 - 2t) = 0$

$18 + 3t + 54 + 12t - 18 + 12t = 0$

$27t + 54 = 0$

$t = -2$

so $\overrightarrow{OP} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$, P is (4, 5, 7)

c $|\overrightarrow{OP}| = \sqrt{4^2 + 5^2 + 7^2} = 3\sqrt{10}$

8 **a**

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (3 - 2)\mathbf{i} + (-2 - (-1))\,\mathbf{j} + (-1 - 2)\mathbf{k}$

$= \mathbf{i} - \mathbf{j} - 3\mathbf{k}$

line is $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ for $\lambda \in \mathbb{R}$

b $2 + \lambda = 7 + 2s \qquad (1)$

$-1 - \lambda = s \qquad (2)$

$2 - 3\lambda = 3 + 2s \qquad (3)$

sub (2) in (1) $\Rightarrow 2 + \lambda = 7 + 2(-1 - \lambda)$

$2 + \lambda = 5 - 2\lambda$

$3\lambda = 3 \Rightarrow \lambda = 1$

$(2) \Rightarrow s = -1 - 1 = -2$

$(3) \Rightarrow 2 = -3(1) - 1 = 3 + 2(-2)$

lines intersect.

point is $(2+1)\mathbf{i}+(-1-1)\mathbf{j}+(2-3)\mathbf{k}$

$3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ ie $(3, -2, -1)$

c $\mathbf{a}-\mathbf{c}=(2-3)\mathbf{i}+(-1-(-2))\,\mathbf{j}+(2-(-1))\mathbf{k}$

$=-\mathbf{i}+\mathbf{j}+3\mathbf{k}$

$|AC|=\sqrt{1+1+3^2}=\sqrt{11}$

d Take dot product of direction vectors:

$(\mathbf{i}-\mathbf{j}-3\mathbf{k}).(2\mathbf{i}+\mathbf{j}+2\mathbf{k})=2-1-6=-5$

Then $-5=\sqrt{11}\sqrt{9}\cos\theta$

$\theta=\cos^{-1}\left(\frac{-5}{3\sqrt{11}}\right)=-120°$ (nearest degree)

Exercise 12L

1 a Position of ship relative to buoy is

$\begin{pmatrix}60\\30\end{pmatrix}-\begin{pmatrix}45\\20\end{pmatrix}=\begin{pmatrix}15\\10\end{pmatrix}$ ie 10Km North, 15Km East

b $\left|\begin{pmatrix}15\\10\end{pmatrix}\right|=\sqrt{15^2+10^2}=5\sqrt{13}$ km

2 a velocity $=\frac{\text{displacement}}{\text{time}}=\begin{pmatrix}\frac{20}{4}\\\frac{-8}{4}\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}\text{ms}^{-1}$

b $\mathbf{s}(t)=\begin{pmatrix}20\\-8\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix}$

$=\begin{pmatrix}20\\-8\end{pmatrix}+6\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}50\\-20\end{pmatrix}\text{m}$

c speed $=|v(t)|=\sqrt{12^2+5^2}=13$ ms^{-1}

d $\mathbf{s}(t)=(4\mathbf{i}-\mathbf{j})+t(12\mathbf{i}-5\mathbf{j})$

$\mathbf{s}(3)=(4\mathbf{i}-\mathbf{j})+3(12\mathbf{i}-5\mathbf{j})=40\mathbf{i}-16\mathbf{j}$

distance $=\sqrt{40^2+16^2}=8\sqrt{29}$ m

e want $\begin{pmatrix}20\\-8\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}+s\begin{pmatrix}12\\-5\end{pmatrix}$

for collision

$20+5t=4+12s$ (1)

$-8-2t=-1-5s$ (2)

$(2)\Rightarrow -7+5s=2t\Rightarrow t=\frac{1}{2}(5s-7)$

$(1)\Rightarrow 20+\frac{5}{2}(5s-7)=4+12s$

$40+25s-35=8+24s$

$s=3$

$(2)\Rightarrow t=\frac{1}{2}(5\times3-7)=4$

When $t=4$, LS $=\begin{pmatrix}20\\-8\end{pmatrix}+\begin{pmatrix}20\\-8\end{pmatrix}=\begin{pmatrix}40\\-16\end{pmatrix}$

When $s=3$, RS $=\begin{pmatrix}4\\-8\end{pmatrix}+3\begin{pmatrix}12\\-5\end{pmatrix}=\begin{pmatrix}40\\-16\end{pmatrix}$

Hence particles will collide.

3 a Let A's position be given by

$\mathbf{a}=(3\mathbf{i}+3\mathbf{j})+t(4\mathbf{i}+3\mathbf{j})$

Let B's position be given by

$\mathbf{b}=(4\mathbf{i}+3\mathbf{j})+s(3\mathbf{i}+3\mathbf{j})$

want to find when $\mathbf{a}=\mathbf{b}$.

$3+4t=4+3s$ (1)

$3+3t=3+3s$ (2)

$(2)\Rightarrow t=s$

$(1)\Rightarrow s=t=1$

They collide 1 hour after 3 pm, ie at 4 pm.

b Collide at $(3\mathbf{i}+3\mathbf{j})+1(4\mathbf{i}+3\mathbf{j})=7\mathbf{i}+6\mathbf{j}$

4 a $\mathbf{r}_x=\begin{pmatrix}11\\3\\-3\end{pmatrix}+t\begin{pmatrix}1\\-1\\4\end{pmatrix}$ $\mathbf{r}_y=\begin{pmatrix}1\\-7\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\9\end{pmatrix}$

$|\mathbf{V}_x|=\sqrt{1^2+(-1)^2+4^2}=\sqrt{18}=3\sqrt{2}$ ms^{-1}

$|\mathbf{V}_y|=\sqrt{2^2+1^2+9^2}=\sqrt{86}$ ms^{-1}

b Meet if $\mathbf{r}_x=\mathbf{r}_y$ at the same time

$\begin{pmatrix}11\\3\\-3\end{pmatrix}+t\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\-7\\-2\end{pmatrix}+s\begin{pmatrix}2\\1\\9\end{pmatrix}$

$11+t=1+2s$ (1)

$3-t=-7+s$ (2)

$-3+4t=-2+9s$ (3)

$(1)\Rightarrow t=2s-10$

$(2)\Rightarrow 3-(2s-10)=-7+s$

$13-2s=-7+s$

$20=3s,\ s=\frac{20}{3}$

$(1)\Rightarrow t=2\left(\frac{20}{3}\right)-10=\frac{10}{3}\neq s$

so ships do not collide.

c $\mathbf{r}_x(10) = \begin{pmatrix} 11 \\ 3 \\ -3 \end{pmatrix} + 10\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ -7 \\ 37 \end{pmatrix}$

$\mathbf{r}_y(10) = \begin{pmatrix} 1 \\ -7 \\ -2 \end{pmatrix} + 10\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ 88 \end{pmatrix}$

$\mathbf{r}_y - \mathbf{r}_x = \begin{pmatrix} 0 \\ 10 \\ 51 \end{pmatrix},$

$|\mathbf{r}_y - \mathbf{r}_x|$

$= \sqrt{10^2 + 51^2} = \sqrt{2701} \approx 51.97\text{m}$

Review exercise

1 $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$

Now $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix} = \frac{-1}{3}\overrightarrow{BC}$

Since they contain a common point (B), A, B, C are collinear.

2 The sides of the triangle are given by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$

$\overrightarrow{AB} = (2\text{I} + 2\text{J}) - (5\text{i} - \text{j} + 6\text{k})$

$= -3\text{i} + 3\text{j} - 6\text{k}$

$\overrightarrow{AC} = (-3\text{i} - 5\text{j} + 8\text{k}) - (5\text{i} - \text{j} + 6\text{k})$

$= -8\text{i} - 4\text{j} + 2\text{k}$

Now $\overrightarrow{AB} \cdot \overrightarrow{AC} = (-3\text{i} + 3\text{j} - 6\text{k}) \cdot (-8\text{i} - 4\text{j} + 2\text{k})$

$= (-3 \times -8) + (3 \times -4) + (-6 \times 2)$

$= +24 - 12 - 12$

$= 0$

Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, the vectors are perpendicular. Hence, A, B, and C form a right-angled triangle.

3 $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5+1 \\ -1+3 \\ -3+(-5) \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix}$

$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 5-1 \\ -1-3 \\ -3-(-5) \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$

$(\mathbf{a} - \mathbf{b}).(\mathbf{a} + \mathbf{b}) = \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix} . \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = 24 - 8 - 16 = 0$

$\Rightarrow \mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ are perpendicular.

4 We need s & t so that

$7s = 3 + 2t$ (1)

$6 + 3s = 1 + 4t$ (2)

$-1 + s = 2 - t$ (3)

$(3) \Rightarrow s = 3 - t$

$(1) \Rightarrow 7(3 - t) = 3 + 2t$

$21 - 7t = 3 + 2t$

$18 = 9t$

$t = 2$

$(3) \Rightarrow s = 3 - 2 = 1$

check in (2) $\Rightarrow 6 + 3(1) = 1 + 4(2)$

LS = RS = 9 so s and t exist.

so P has position vector $\begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 0 \end{pmatrix}$

Point (7, 9, 0)

5 a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

b $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} . \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -3 - 6 = -9$

c $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}| \cos B\hat{A}C$

$-9 = \sqrt{3^2 + 3^2}\sqrt{(-1)^2 + (-2)^2} \cos B\hat{A}C$

$-9 = \sqrt{18}\sqrt{5} \cos B\hat{A}C$

$-9 = 3\sqrt{2}\sqrt{5} \cos B\hat{A}C$

$\cos B\hat{A}C = \frac{-3}{\sqrt{2}\sqrt{5}}$

6 a P has position vector

$\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-8 \\ 2+8 \\ -3+4 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix}$

P is $(-2, 10, 1)$

b Suppose $\begin{pmatrix}-2\\10\\1\end{pmatrix}=\begin{pmatrix}0\\-12\\7\end{pmatrix}+t\begin{pmatrix}-1\\11\\-3\end{pmatrix}$ for some t

$-2=-t \qquad (1)$

$10=-12+11t \qquad (2)$

$1=7-3t \qquad (3)$

$(1)\Rightarrow t=2$

$(2)\Rightarrow 10=-12+11(2)=10$

$(3)\Rightarrow 1=7-3(2)=7-6=1$

so equations are consistent.

so $t=2$ gives $\begin{pmatrix}-2\\10\\1\end{pmatrix}$ $\therefore P$ lies on L_2

7 **a** L_2: $\mathbf{r}=\begin{pmatrix}2\\2\\4\end{pmatrix}+s\begin{pmatrix}1\\3\\2\end{pmatrix}$

b $0=\begin{pmatrix}1\\7\\3\end{pmatrix}\cdot\begin{pmatrix}4\\x\\1\end{pmatrix}=4+7x+3=7+7x\Rightarrow x=-1$

c $\begin{pmatrix}2\\-3\\-3\end{pmatrix}+t\begin{pmatrix}1\\7\\3\end{pmatrix}=\begin{pmatrix}7\\5\\1\end{pmatrix}+q\begin{pmatrix}4\\-1\\1\end{pmatrix}$

$2+t=7+4q \qquad (1)$

$-3+7t=5-q \qquad (2)$

$-3+3t=1+q \qquad (3)$

8 Suppose $\mathbf{r}_1=\mathbf{r}_2$

Then $\begin{pmatrix}-4\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}+\mu\begin{pmatrix}-12\\5\end{pmatrix}$

$-4+4\lambda=4-12\mu \qquad (1)$

$3+17\lambda=9+5\mu \qquad (2)$

$(1)\Rightarrow 4\lambda=8-12\mu$

$\lambda=2-3\mu$

$(2)\Rightarrow 17(2-3\mu)=6+5\mu$

$34-51\mu=6+5\mu$

$28=56\mu$

$\mu=\frac{1}{2}$

$(1)\Rightarrow \lambda=2-\frac{3}{2}=\frac{1}{2}$

So ships collide after $\frac{1}{2}$ hour, ie 12.30 pm.

collide at $\begin{pmatrix}-4\\3\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix}$

b At 12.15, A has position $\begin{pmatrix}-4\\3\end{pmatrix}+\frac{1}{4}\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}$

so after 12:15, A's position given by

$\mathbf{r}_1=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}+t\begin{pmatrix}16\\17\end{pmatrix}$ where t is time after 12:15

At 12.30, A's position is

$\mathbf{r}_1=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}+\frac{1}{4}\begin{pmatrix}16\\17\end{pmatrix}=\begin{pmatrix}1\\\frac{23}{2}\end{pmatrix}$

Distance is $\begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix}-\begin{pmatrix}1\\\frac{23}{2}\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}$

so ships are 3km apart.

Review exercise – GDC

1 $\begin{pmatrix}3\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-4\end{pmatrix}=\left|\begin{pmatrix}3\\5\end{pmatrix}\right|\left|\begin{pmatrix}2\\-4\end{pmatrix}\right|\cos A$

$6-20=\sqrt{3^2+5^2}\sqrt{2^2+(-4)^2}\cos A$

$\frac{-14}{\sqrt{34}\sqrt{20}}=\cos A$

$A=122.4\approx 122°$

2 **a** $\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}$

$=\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\5\end{pmatrix}$

$\overrightarrow{QP}=\overrightarrow{OP}-\overrightarrow{OQ}$

$=\begin{pmatrix}3\\-2\\1\end{pmatrix}-\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix}$

b $\begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}=\left|\begin{pmatrix}-1\\0\\5\end{pmatrix}\right|\left|\begin{pmatrix}0\\-1\\1\end{pmatrix}\right|\cos P\hat{Q}R$

$0+0+5=\sqrt{(-1)^2+0^2+5^2}$

$\sqrt{0^2+(-1)^2+1^2}\cos P\hat{Q}R$

$\frac{5}{\sqrt{26}\sqrt{2}}=\cos P\hat{Q}R$

$=46.1\approx 46°$

c $\frac{1}{2}|\overrightarrow{QR}||\overrightarrow{QP}|\sin 46=\text{Area}$

$=\frac{1}{2}\sqrt{52}\sin 46$

$=2.60\text{ units}^2$

3 **a** **i** $\overrightarrow{OC} = 4\mathbf{j}$

ii $\overrightarrow{OB} = \mathbf{i} + \sqrt{2^2 - 1^2}\,\mathbf{k} = \mathbf{i} + \sqrt{3}\,\mathbf{k}$

iii $\overrightarrow{OD} = 2\mathbf{i} + 4\mathbf{j}$

b **i** $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\mathbf{i} - \sqrt{3}\mathbf{k} + 4\mathbf{j} = -\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}$

ii $\overrightarrow{BD} = \mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}$

c **i** $|\overrightarrow{BC}| = \sqrt{1^2 + 16 + 3} = \sqrt{20} = 2\sqrt{5}$

ii $|\overrightarrow{BD}| = \sqrt{1^2 + 16 + 3} = \sqrt{20} = 2\sqrt{5}$

iii $(-\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k})\cdot(\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}) =$

$-1 + 16 + 3 = 18$

$18 = 2\sqrt{5} \times 2\sqrt{5} \cos D\hat{B}C$

$\frac{18}{20} = \frac{9}{10} \cos D\hat{B}C$

$25.8° = D\hat{B}C$

4 **a** If perpendicular, $\mathbf{a}\quad\mathbf{b} = 0$

$(x\mathbf{i} + (x-2)\mathbf{j} + \mathbf{k})\quad(x^2\mathbf{i} - 2x\mathbf{j} - 12x\mathbf{k}) = 0$

$x^3 - 2x(x-2) - 12x = 0$

$x^3 - 2x^2 + 4x - 12x = 0$

$x(x^2 - 2x - 8) = 0$

$x(x-4)(x+2) = 0$

$x = 0,\ x = 4,\ \text{or } x = -2$

b Let $x = -1$. $\mathbf{a} = -1\mathbf{i} + -3\mathbf{j} + \mathbf{k},\ \ \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 12\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos C$

$-1 + -6 + 12 = \sqrt{(-1)^2 + (-3)^2 + 1^2}$

$\sqrt{1^2 + 2^2 + 12^2} \cos C$

$5 = \sqrt{11}\sqrt{149} \cos C$

$\frac{5}{\sqrt{11}\sqrt{149}} = \cos C$

$82.9° = C$

5 **a** $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \begin{pmatrix}1\\5\\5\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\6\\2\end{pmatrix}$$

$$\overrightarrow{OP}.\overrightarrow{PQ} = \begin{pmatrix}1\\-1\\3\end{pmatrix}.\begin{pmatrix}0\\6\\2\end{pmatrix} = 0 - 6 + 6 = 0$$

$\therefore \overrightarrow{OP}$ is perpendicular to $\overrightarrow{PQ}$

b $$\mathbf{r}_1 = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\6\\2\end{pmatrix}$$

c If intersect, $r_1 = r_2$

$$\begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\6\\2\end{pmatrix} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$1 = 2 + \mu \qquad \mu = -1$

$-1 + 6\lambda = -1 - 3\mu \qquad 6\lambda = -3\mu = 3 \qquad \lambda = \frac{1}{2}$

$3 + 2\lambda = 2 - 2\mu \qquad \text{LHS } 3 + 2 \times \frac{1}{2} = 4$

$\text{RHS } 2 - 2 \times -1 = 2 + 2 = 4$

consistent values for λ and μ in all 3 equations

lines intersect

Let $\lambda = \frac{1}{2}$ in r_1

$$\text{Position vector} = \begin{pmatrix}1+0\\-1+3\\3+1\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}$$

d

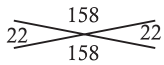
$$\begin{pmatrix}0\\6\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\-2\end{pmatrix} = \left|\begin{pmatrix}0\\6\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\-3\\-2\end{pmatrix}\right| \cos A$$

$-18 + -4 = \sqrt{0^2 + 6^2 + 2^2}\ \sqrt{1^2 + (-3)^2 + (-2)^2} \cos A$

$-22 = \sqrt{40}\sqrt{14} \cos A$

$\frac{-22}{\sqrt{40}\sqrt{14}} = \cos A$

$A = 158°$



acute angle between lines is 22°

6 **a** $t = 0 \quad A = \begin{pmatrix}0\\0\\6\end{pmatrix}$

$t = 2 \quad B = \begin{pmatrix}6\\-2\\6\end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \begin{pmatrix}6\\-2\\6\end{pmatrix} - \begin{pmatrix}0\\0\\6\end{pmatrix} = \begin{pmatrix}6\\-2\\0\end{pmatrix}$$

b Position vector = initial position + t (directional vector of $\overrightarrow{AB}$)

$$= \text{initial position} + t\begin{pmatrix}3\\-1\\0\end{pmatrix}$$

c (36, 18, 0)

d

$$v = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}, \text{ speed} = |v| = \sqrt{(-3)^2 + (-4)^2 + 1^2}$$

$$= \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$= 5.10 \text{ ms}^{-1}$$

e

$$\begin{pmatrix} 36 \\ 18 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$36 - 3t = 3s \quad 36 - 18 = 18 \qquad s = 6$

$18 - 4t = -s \quad 18 - 24 = -s \qquad s = 6$ consistent

$t = 6$

$t = 6$ seconds

f $c = (36 - 18, 18 - 24, 0 + 6)$

$= (18, -6, 6)$

13 Circular functions

Answers

Skills check

You should know these values,without using your GDC.

1 **a** $\frac{\sqrt{2}}{2}$ **b** $\sqrt{3}$

c $-\frac{\sqrt{3}}{2}$ **d** $-\frac{\sqrt{2}}{2}$

2 **a** $\frac{\sqrt{3}}{2}$ **b** -1

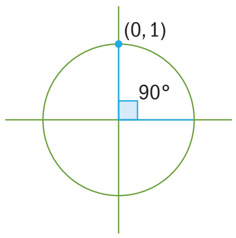
c -1 **d** -0.5

3 **a** -1.48 **b** ± 2

4 **a** $-0.182, 2.40$ **b** ± 1.14

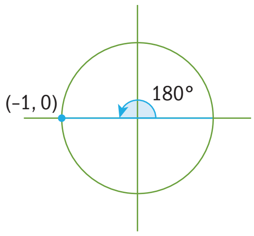
Investigation

1



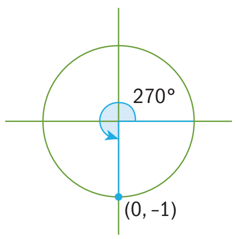
$\sin 90° = 1$, $\cos 90° = 0$, $\tan 90°$ does not exist

2



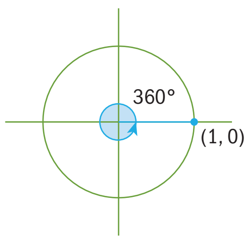
$\sin 180° = 0$, $\cos 180° = -1$, $\tan 180° = 0$

3



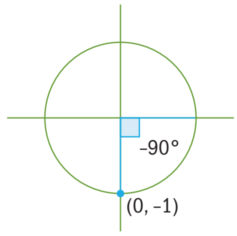
$\sin 270° = -1$, $\cos 270° = 0$, $\tan 270°$ does not exist

4



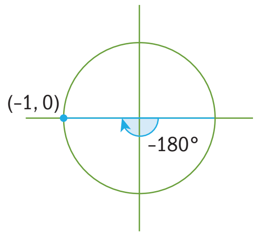
$\sin 360° = 0$, $\cos 360° = 1$, $\tan 360° = 0$

5



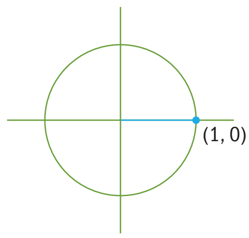
$\sin(-90°) = -1$, $\cos(-90°) = 0$, $\tan(-90°)$ does not exist

6



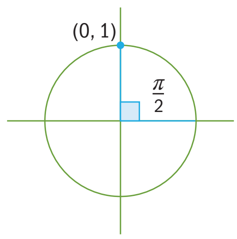
$\sin(-180°) = 0$, $\cos(-180°) = -1$, $\tan(-180°) = 0$

7



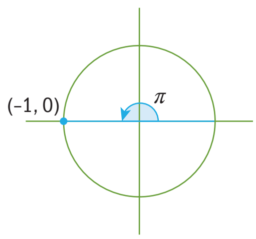
$\sin 0 = 0$, $\cos 0 = 1$, $\tan 0 = 0$

8



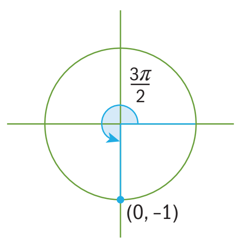
$\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\tan\frac{\pi}{2}$ does not exist

9



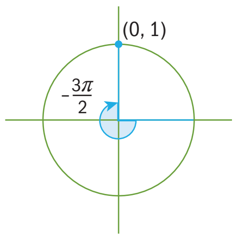
$\sin\pi = 0$, $\cos\pi = -1$, $\tan\pi = 0$

10



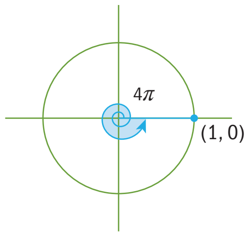
$\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, $\tan\frac{3\pi}{2}$ does not exist

11



$\sin\left(-\frac{3\pi}{2}\right) = 1$, $\cos\left(-\frac{3\pi}{2}\right) = 0$, $\tan -\frac{3\pi}{2}$ does not exist

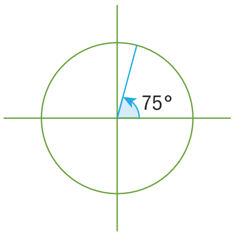
12



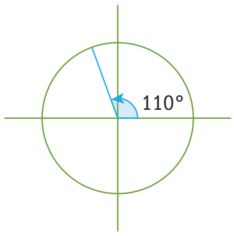
$\sin 4\pi = 0$, $\cos 4\pi = 1$, $\tan 4\pi = 0$

Exercise 13A

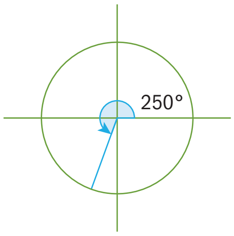
1 a



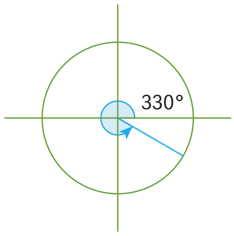
b



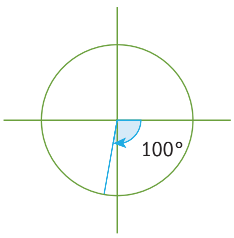
c



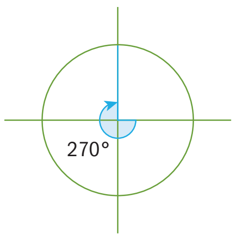
d



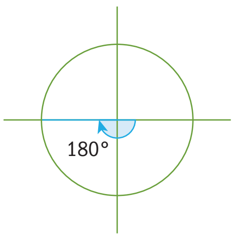
e



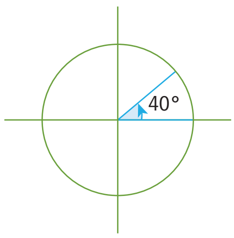
f



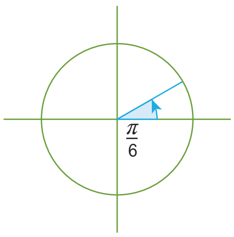
g



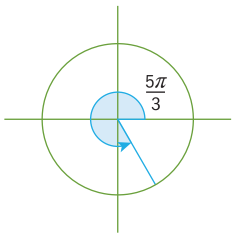
h



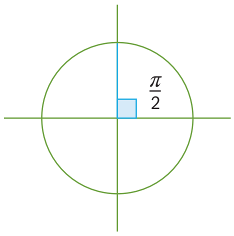
2 a



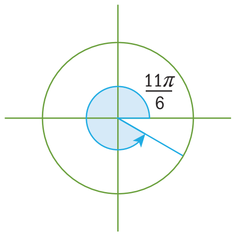
b



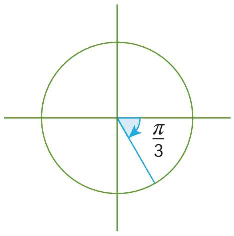
c



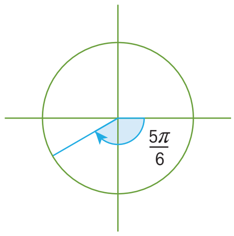
d



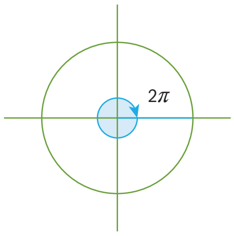
e



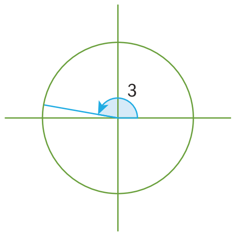
f



g

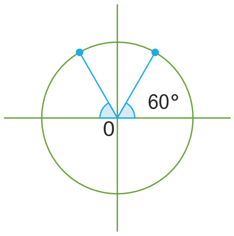


h



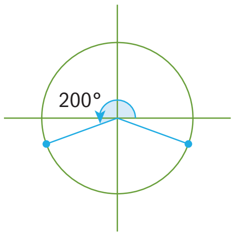
For questions 3–8, there are many other possible correct answers.

3 a



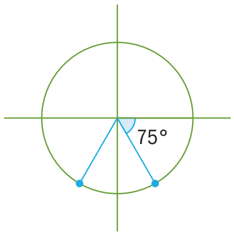
120°, –240°, –300°

b



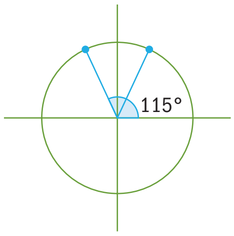
340°, –20°, –160°

c



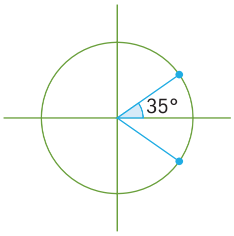
255°, 285°, –105°

d



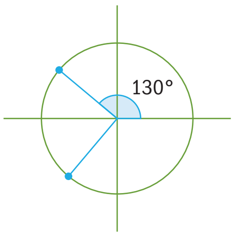
65°, –245°, –295°

4 **a**



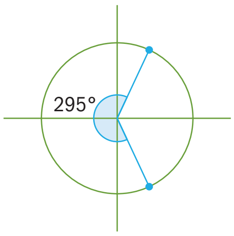
$-35°, \pm 325°$

b



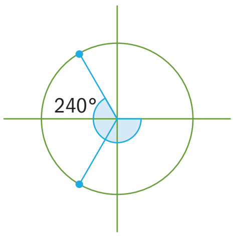
$-130°, \pm 230°$

c



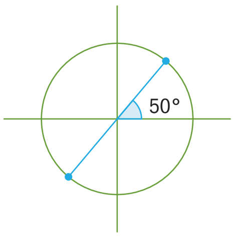
$-295°, \pm 65°$

d



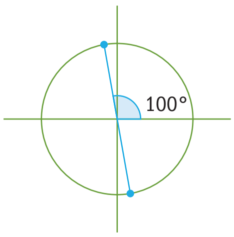
$240°, \pm 120°$

5 **a**



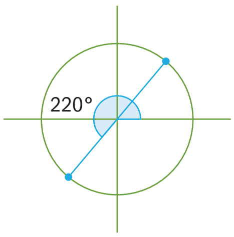
$230°, -130°, -310°$

b



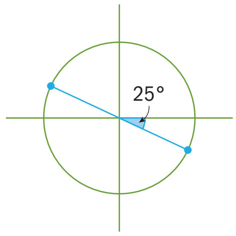
$280°, -80°, -260°$

c



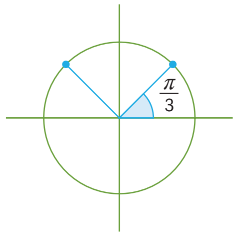
$40°, -140°, -320°$

d



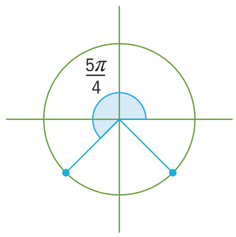
$155°, 335°, -205°$

6 **a**



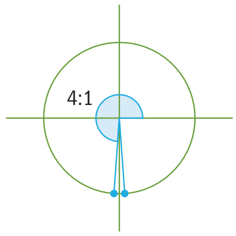
$\frac{2\pi}{3}, -\frac{4\pi}{3}, -\frac{5\pi}{3}$

b



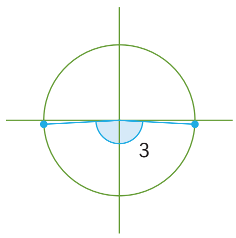
$\frac{7\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}$

c



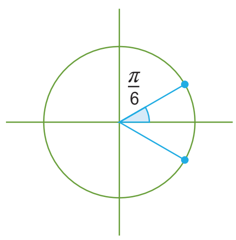
$3\pi - 4.1, 4.1 - 2\pi, \pi - 4.1$

d



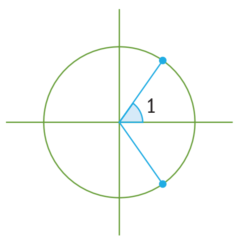
$\pi + 3, 2\pi - 3, 3 - \pi$

7 **a**



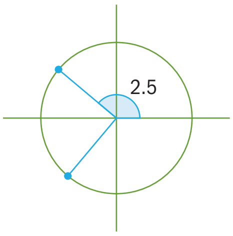
$-\frac{\pi}{6}, \pm \frac{11\pi}{6}$

b



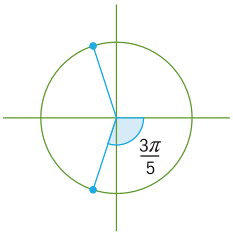
$-1, \pm(1 - 2\pi)$

c



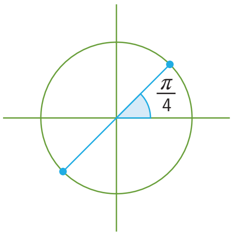
$-2.5, \pm(2.5 - 2\pi)$

d



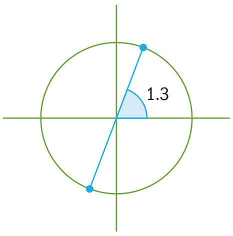
$\frac{3\pi}{5}, \pm\frac{7\pi}{5}$

8 a



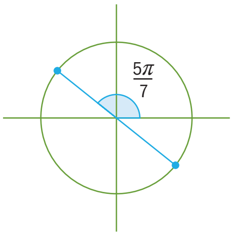
$\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{7\pi}{4}$

b



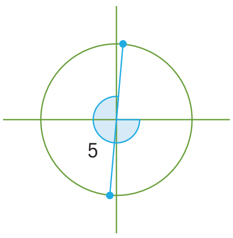
$1.3 + \pi, 1.3 - \pi, 1.3 - 2\pi$

c



$\frac{12\pi}{7}, -\frac{2\pi}{7}, -\frac{9\pi}{7}$

d



$2\pi - 5, \pi - 5, -5 - \pi$

Exercise 13B

1 a $\sin 110 = \sin 70 = 0.940$

b $\cos(-70) = \cos 70 = 0.342$

c $\cos 250 = -\cos 70 = -0.342$

d $\sin 290 = -\sin 70 = -0.940$

2 a $\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

b $\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

c $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

d $\cos\left(-\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

3 a $\sin(180 - A) = \sin(A) = 0.8$

b $\cos(-A) = \cos(A) = 0.6$

c $\cos(360 - A) = \cos(A) = 0.6$

d $\sin(180 + A) = -\sin(A) = -0.8$

e $\tan(A) = \frac{\sin(A)}{\cos(A)} = \frac{0.8}{0.6} = \frac{4}{3}$

f $\tan(-A) = -\tan(A) = -\frac{4}{3}$

g $\sin(360 - A) = -\sin(A) = -0.8$

h $\tan(180 + A) = \tan(A) = \frac{4}{3}$

4 a $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$

b $\sin(\pi - \theta) = \sin\theta = a$

c $\cos(\pi + \theta) = -\cos\theta = -b$

d $\tan(\pi + \theta) = \tan\theta = \frac{a}{b}$

e $\sin(\pi + \theta) = -\sin\theta = -a$

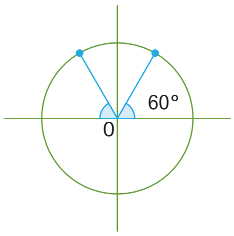
f $\cos(-\theta) = \cos\theta = b$

g $\sin(2\pi - \theta) = -\sin\theta = -a$

h $\cos(\theta - \pi) = -\cos\theta = -b$

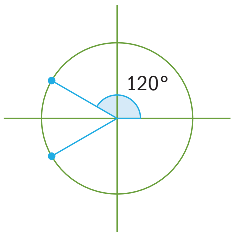
Exercise 13C

1 a



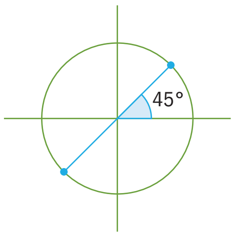
$x = -300°, -240°, 60°, 120°$

b



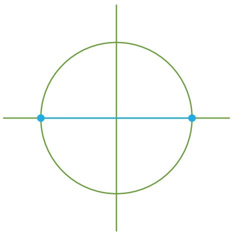
$x = \pm 120°, \pm 240°$

c



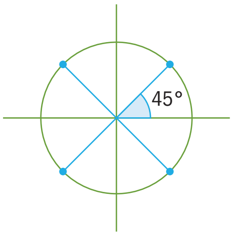
$x = -315°, -135°, 45°, 225°$

d



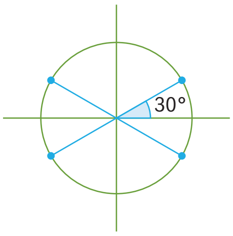
$x = -360°, -180°, 0°, 180°, 360°$

e $\cos x = \pm\frac{1}{\sqrt{2}}$



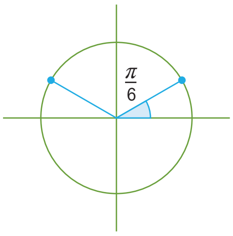
$x = \pm 45°, \pm 135°, \pm 225°, \pm 315°$

f $\tan x = \pm\frac{1}{\sqrt{3}}$



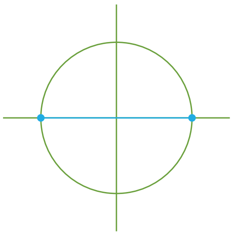
$x = \pm 30°, \pm 150°, \pm 210°, \pm 330°$

2 a



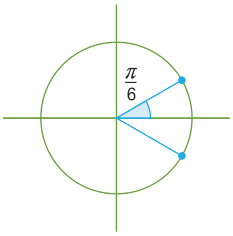
$\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

b



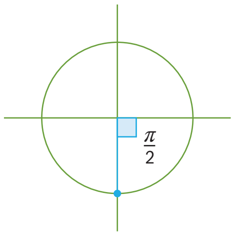
$\theta = 0, \pm\pi, \pm 2\pi$

c



$\theta = \pm\frac{\pi}{6}, \pm\frac{11\pi}{6}$

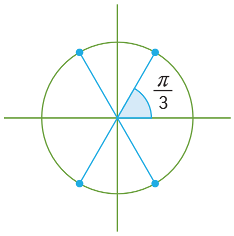
d



$\theta = -\frac{\pi}{2}, \frac{3\pi}{2}$

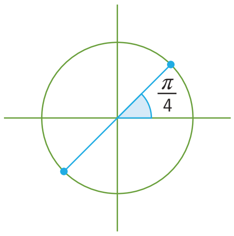
e $\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$



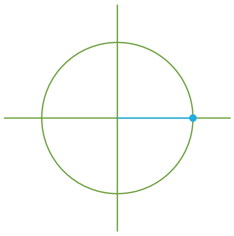
$\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}, \pm\frac{5\pi}{3}$

f $\tan\theta = 1$



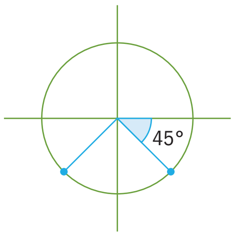
$\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

3 a



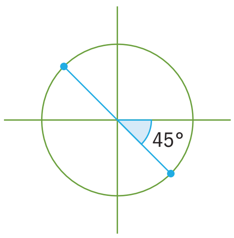
$\theta = 0°, 360°, 720°$

b



$\theta = -135°, -45°, 225°, 315°, 585°, 675°$

c $\tan\,\theta = -1$

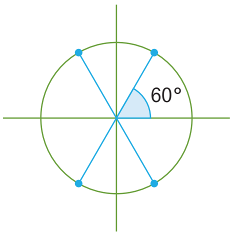


$\theta = -225°, -45°, 135°, 315°, 495°, 675°$

d $3\tan^2\theta = 9$

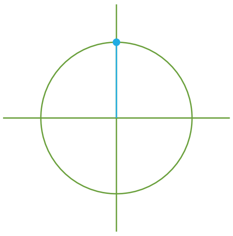
$3\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$



$\theta = \pm 60°, \pm 120°, 240°, 300°, 420°, 480°, 600°, 660°$

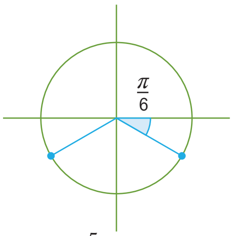
4 a



$x = \frac{\pi}{2}$

b $2\sin x = -1$

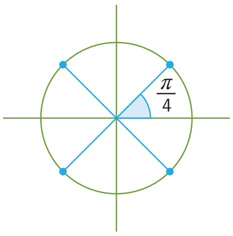
$\sin x = \frac{-1}{2}$



$x = -\frac{5\pi}{6}, -\frac{\pi}{6}$

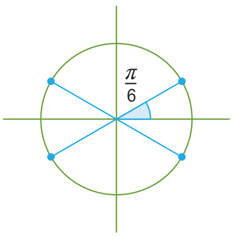
c $\sin^2 x = \frac{5}{10} = \frac{1}{2}$

$\sin x = \pm\frac{1}{\sqrt{2}}$



$x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

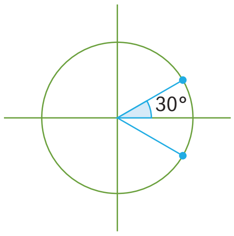
d $4\cos^2 x = 3$ $\quad \cos^2 x = \frac{3}{4}$ $\quad \cos x = \pm\frac{\sqrt{3}}{2}$



$x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

Exercise 13D

1 a

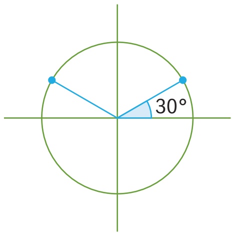


$2x = \pm 30, \pm 330$

$x = \pm 15°, \pm 165°$

b $6\sin(2x) = 3$

$\sin(2x) = \frac{1}{2}$

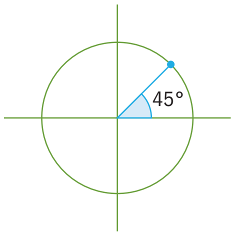


$2x = -330, -210, 30, 150$

$x = -165°, -105°, 15°, 75°$

c $\sin\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$

$\tan\left(\frac{x}{2}\right) = 1$

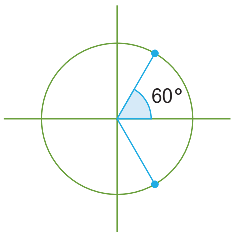


$\frac{x}{2} = 45$

$x = 90°$

d $\tan^2\left(\frac{x}{3}\right) = 3$

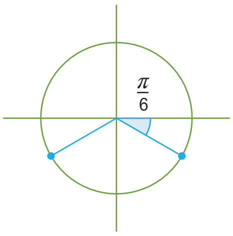
$\tan\left(\frac{x}{3}\right) = \pm\sqrt{3}$



$\frac{x}{3} = \pm 60$

$x = \pm 180°$

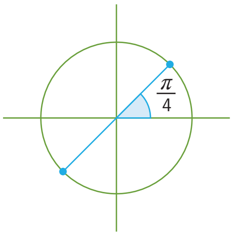
2 **a**



$2\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\theta = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

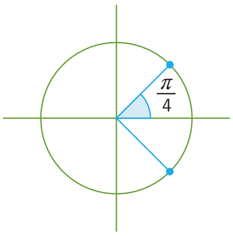
b



$3\theta = -\frac{11\pi}{4} - \frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

$\theta = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

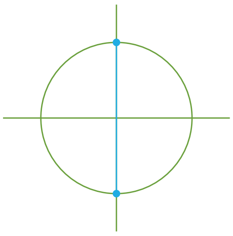
c $\cos\left(\frac{\theta}{2}\right) = \pm\frac{1}{\sqrt{2}}$



$\frac{\theta}{2} = \pm\frac{\pi}{4}$

$\theta = \pm\frac{\pi}{2}$

d $\sin\left(\frac{2\theta}{3}\right) = \pm 1$



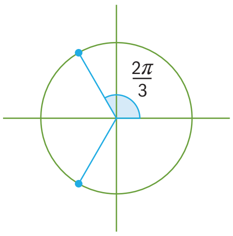
$\frac{2\theta}{3} = \pm\frac{\pi}{2}$

$\theta = \pm\frac{3\pi}{4}$

3 **a** $(2\cos x + 1)(\cos x - 3) = 0$

The second factor gives $\cos x = 3$, which has no solution.

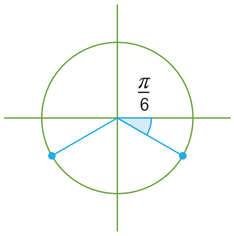
$\cos x = -\frac{1}{2}$



$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

b $(2\sin x + 1)(\sin x + 1) = 0$

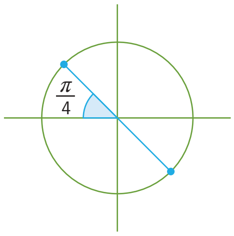
$\sin x = -\frac{1}{2}$ or $\sin x = -1$



$x = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

c $(\tan x + 1)(\tan x + 1) = 0$

$\tan x = -1$



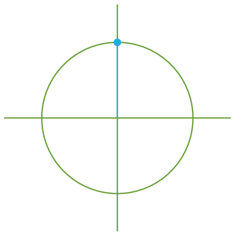
$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

d $\sin^2 x - 6\sin x + 5 = 0$

$(\sin x - 1)(\sin x - 5) = 0$

The second factor gives $\sin x = 5$, which has no solution.

$\sin x - 1 = 0$



$x = \frac{\pi}{2}$

Exercise 13E

1 **a** $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{5}{6}\right)^2 + \cos^2\theta = 1 \rightarrow \cos^2\theta = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$

$\cos\theta = \frac{\sqrt{11}}{6}$

$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{5}{6}\right)\left(\frac{\sqrt{11}}{6}\right) = \frac{5\sqrt{11}}{18}$

b $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{6}\right)^2 = 1 - \frac{50}{36} = -\frac{7}{18}$

c $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{\left(\frac{5\sqrt{11}}{18}\right)}{\left(-\frac{7}{18}\right)} = -\frac{5\sqrt{11}}{7}$

2 **a** $\sin^2 x + \cos^2 x = 1$

$\sin^2 x + \left(-\frac{2}{3}\right)^2 = 1 \rightarrow \sin^2 x = 1 - \left(-\frac{2}{3}\right)^2 = \frac{5}{9}$

$\sin x = \frac{\sqrt{5}}{3}$

$\sin(2x) = 2\sin x\cos x = 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$

b $\cos(2x) = 2\cos^2 x - 1 = 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$

c $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{\left(-\frac{4\sqrt{5}}{9}\right)}{\left(-\frac{1}{9}\right)} = 4\sqrt{5}$

3 **a** $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{5}{6}\right)^2 = 1 \rightarrow \sin^2\theta = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$

$\sin\theta = \frac{\sqrt{11}}{6}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{\sqrt{11}}{6}\right)}{\left(\frac{5}{6}\right)} = \frac{\sqrt{11}}{5}$

b $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{11}}{6}\right)\left(\frac{5}{6}\right) = \frac{5\sqrt{11}}{18}$

c $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{5}{6}\right)^2 - 1 = \frac{50}{36} - 1 = \frac{7}{18}$

d $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{\left(\frac{5\sqrt{11}}{18}\right)}{\left(\frac{7}{18}\right)} = \frac{5\sqrt{11}}{7}$

4 **a** $\sin^2 x + \cos^2 x = 1$

$\left(-\frac{1}{8}\right)^2 + \cos^2 x = 1 \rightarrow \cos^2 x = 1 - \left(-\frac{1}{8}\right)^2 = \frac{63}{64}$

$\cos x = -\frac{\sqrt{63}}{8}$

$\sin(2x) = 2\sin x\cos x = 2\left(-\frac{1}{8}\right)\left(-\frac{\sqrt{63}}{8}\right) = \frac{\sqrt{63}}{32}$

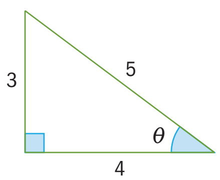
b $\cos(2x) = 1 - 2\sin^2 x = 1 - 2\left(-\frac{1}{8}\right)^2 = 1 - \frac{1}{32} = \frac{31}{32}$

c $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{\left(\frac{\sqrt{63}}{32}\right)}{\left(\frac{31}{32}\right)} = \frac{\sqrt{63}}{31}$

d $\sin(4x) = 2\sin(2x)\cos(2x)$

$= 2\left(\frac{\sqrt{63}}{32}\right)\left(\frac{31}{32}\right) = \frac{31\sqrt{63}}{512}$

5 **a**



$\sin\theta = \frac{3}{5}$

b $\cos\theta = \frac{4}{5}$

c $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

d $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$

6 **a** $\sin^2(2x) + \cos^2(2x) = 1$

$\left(\frac{24}{25}\right)^2 + \cos^2(2x) = 1$

$\rightarrow \cos^2(2x) = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625}$

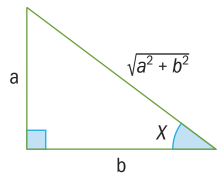
$\cos(2x) = -\frac{7}{25}$

b $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{\left(\frac{24}{25}\right)}{\left(-\frac{7}{25}\right)} = -\frac{24}{7}$

c $\sin(4x) = 2\sin(2x)\cos(2x) = 2\left(\frac{24}{25}\right)\left(-\frac{7}{25}\right) = -\frac{336}{625}$

d $\cos(4x) = 1 - 2\sin^2(2x) = 1 - 2\left(\frac{24}{25}\right)^2 = -\frac{527}{625}$

7 **a**



$\sin x = \frac{a}{\sqrt{a^2 + b^2}}$

b $\cos x = \frac{b}{\sqrt{a^2 + b^2}}$

c $\sin(2x) = 2\sin x\cos x = 2\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$

$= \frac{2ab}{a^2 + b^2}$

d $\cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 - \left(\frac{a}{\sqrt{a^2+b^2}}\right)^2$

$= \frac{b^2 - a^2}{a^2 + b^2}$

Exercise 13F

1 **a** $2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = 90°$ or $x = 30°, 150°$

b $\tan(2x) = 1$

$2x = 45, 225$

$x = 22.5°, 112.5°$

c $\sin x + \cos x = 0$

$\sin x = -\cos x$

$\tan x = -1$

$x = 135°$

d $\cos x = \pm\frac{1}{\sqrt{2}}$

$x = 45°, 135°$

2 **a** $\sin 2x = \frac{\sqrt{3}}{2}$

$2x = -300, -240, 60, 120$

$x = -150°, -120°, 30°, 60°$

b $\sin x - \sin^2 x = \cos^2 x$

$\sin x = \sin^2 x + \cos^2 x = 1$

$x = 90°$

c $\cos^2 x - \sin^2 x = \frac{1}{2}$

$\cos 2x = \frac{1}{2}$

$2x = \pm 60, \pm 300$

$x = \pm 30°, \pm 150°$

d $1 - 2\sin^2 x = \sin x$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0)$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = 30°, 150°$ or $x = -90°$

3 a $\frac{\sin x}{\cos x} = \sin x$

$\sin x = \sin x \cos x$

$\sin x \cos x - \sin x = \sin x(\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = 1$

$x = 0, \pi$

b $\cos 2x = \frac{1}{\sqrt{2}}$

$2x = \frac{\pi}{4}, \frac{7\pi}{4}$

$x = \frac{\pi}{8}, \frac{7\pi}{8}$

c $2\cos^2 x - 1 = \cos x$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

$x = \frac{2\pi}{3}$ or $x = 0$

d $2\sin(2x)\cos(2x) = \sin(2x)$

$2\sin(2x)\cos(2x) - \sin(2x) = 0$

$(\sin(2x))(2\cos(2x) - 1) = 0$

$\sin(2x) = 0$ or $2\cos(2x) - 1 = 0$

$\sin(2x) = 0$ or $\cos(2x) = \frac{1}{2}$

$2x = 0, \pi, 2\pi$ or $2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = 0, \frac{\pi}{2}, \pi$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}$

4 a $\sin^2 2x + 2\sin 2x\cos 2x + \cos^2 2x = 2$

$2\sin 2x\cos 2x + 1 = 2$

$\sin 4x = 1$

$4x = \frac{\pi}{2}, \frac{5\pi}{2}$

$x = \frac{\pi}{8}, \frac{5\pi}{8}$

b $\sin x - 1 = 1 - \sin^2 x$

$\sin^2 x + \sin x - 2 = 0$

$(\sin x - 1)(\sin x + 2) = 0$

$\sin x = 1$ or $\sin x = -2$ which is invalid

$x = \frac{\pi}{2}$

c $\cos^2 x = 2\cos^2 x - 1$

$\cos^2 x = 1$

$\cos x = \pm 1$

$x = 0, \pi$

d $\sin^2 x = \frac{1}{2}$

$\sin x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

5 *working may vary*

a $\sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + \sin(2x)$

LHS: $1 + 2\sin x\cos x = 1 + \sin(2x)$ RHS

b $\frac{1}{\cos\theta} = \sin\theta\frac{\sin\theta}{\cos\theta} + \cos\theta$

$\frac{1}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} + \cos\theta$

$1 = \sin^2\theta + \cos^2\theta$

c $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$\cos 2x \times 1 = \cos 2x$

6 $2\sin 3x\cos 3x = \sin 2(3x)$

$= \sin 6x$

$\Rightarrow k = 6$

7 $\cos 4x = \cos 2(2x)$

$= 1 - 2\sin^2(2x)$

$= 1 - 2(\sin(2x))^2$

$= 1 - 2(2\sin x\cos x)^2$

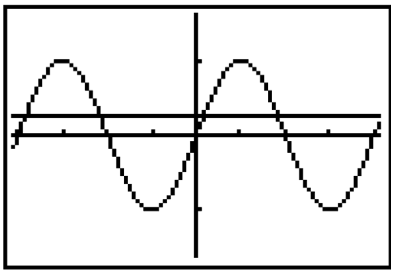
$= 1 - 2(4\sin^2 x\cos^2 x)$

$= 1 - 8\sin^2 x\cos^2 x$

$\Rightarrow b = 8$

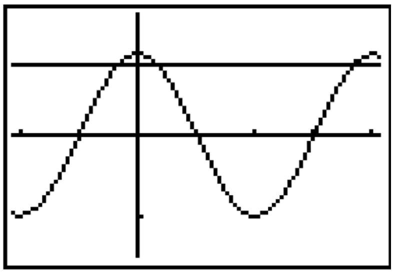
Exercise 13G

1



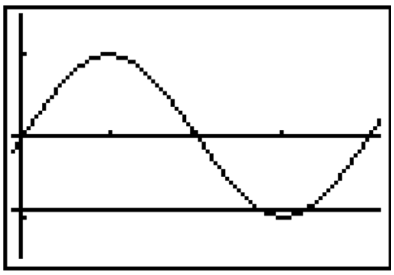
$-346°, -194°, 14°, 166°$

2



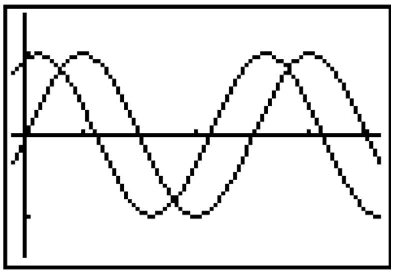
$\pm 27°, 333°$

3



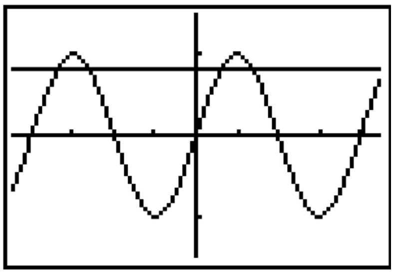
$244°, 296°$

4



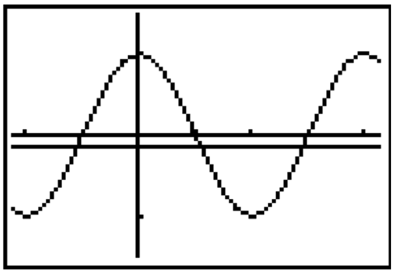
55°, 235°, 415°

5



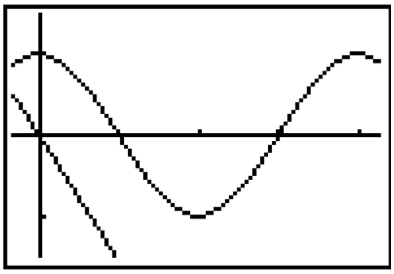
−5.33, −4.10, 0.955, 2.19

6



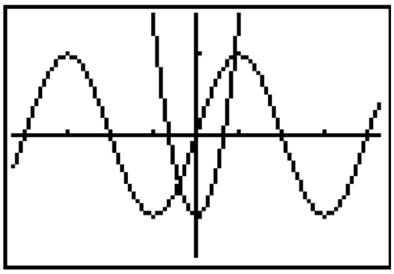
±1.71, 4.58

7



−0.739

8



−0.637, 1.41

Investigation: graphing tan *x*

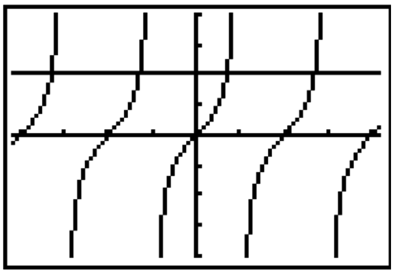
1

Angle measure (x) (degrees)	**Tangent value ($\tan x$)**
0	0
-30, +30	$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
-45, +45	-1, 1
-60, +60	$-\sqrt{3}$, $\sqrt{3}$
120	$-\sqrt{3}$
135	-1
150	$-\frac{1}{\sqrt{3}}$
180	0
210	$\frac{1}{\sqrt{3}}$
225	1
240	$\sqrt{3}$
300	$-\sqrt{3}$
315	-1
330	$-\frac{1}{\sqrt{3}}$
360	0

3 tan ±90° and tan ±270° are undefined. The limit of the tangent as the angle approaches ±90° or ±270° is infinite. Asymptotes are often shown on graphs for values that do not exist.

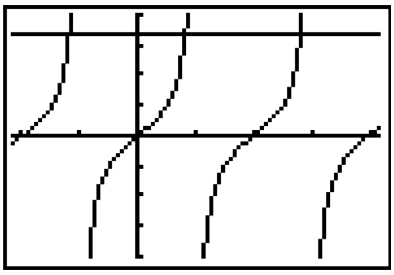
Exercise 13H

1



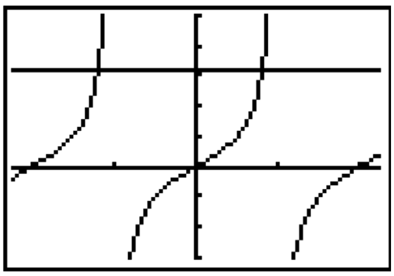
−297°, −117°, 63°, 243°

2



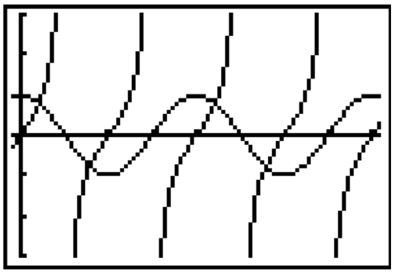
−107°, 73°, 253°

3



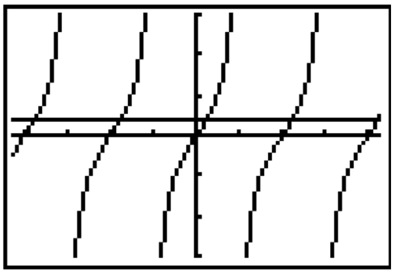
124° , 304°

4



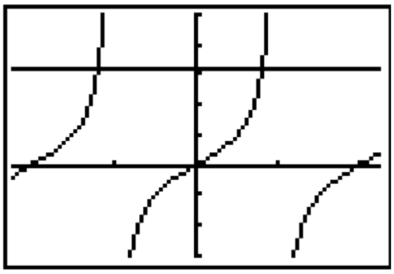
38°, 142°, 398°, 502°

5



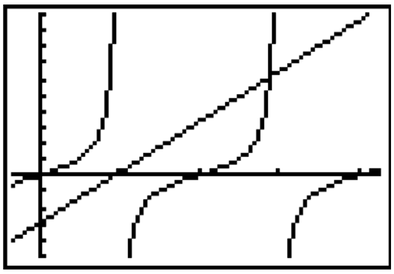
−5.88, −2.74, 0.405, 3.55

6



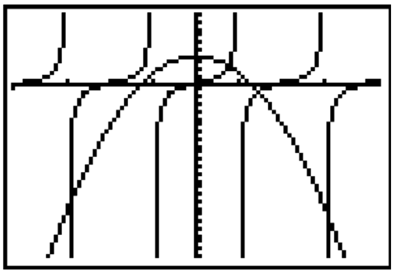
−1.88, 1.26

7



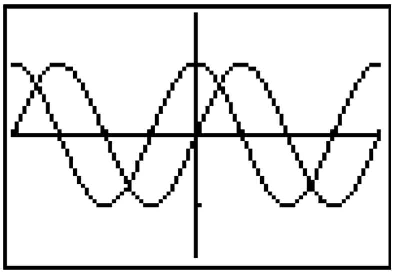
4.55

8

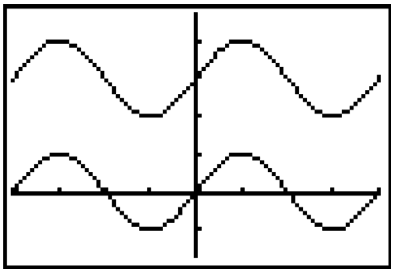


−4.66, 1.20, 2.28, 4.77

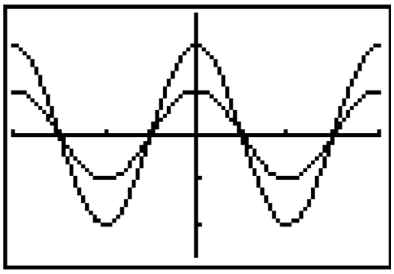
Investigation – transformations of sin*x* and cos*x*



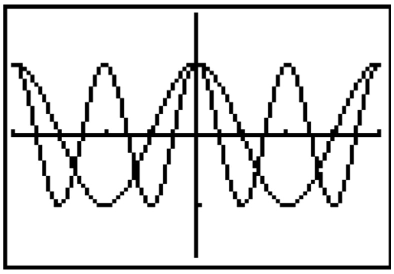
1



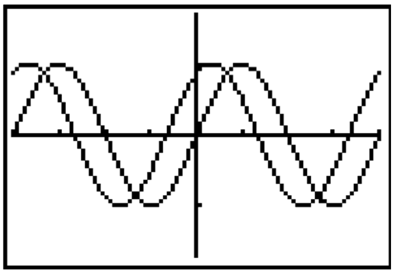
2



3



4



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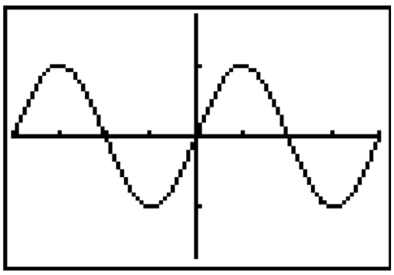
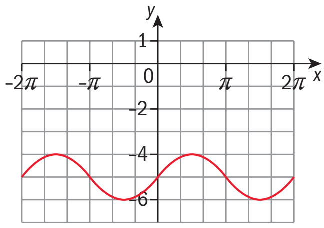


Exercise 13I

1



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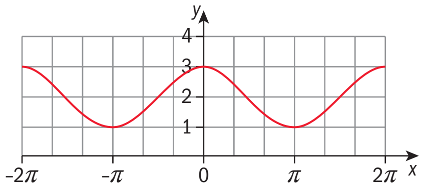

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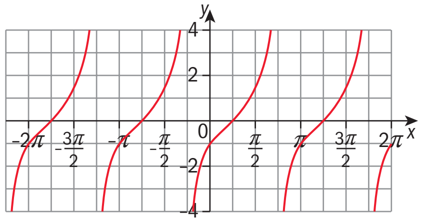

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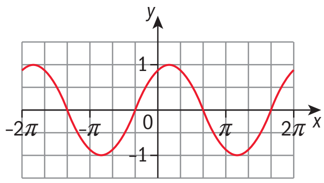

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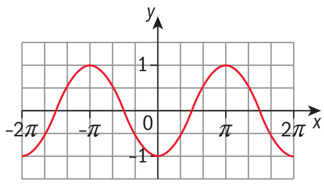

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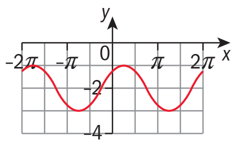

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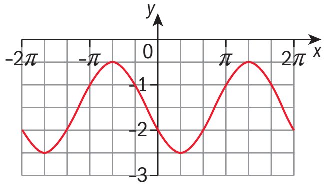

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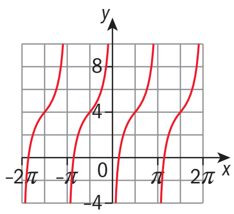


for questions 9 – 12, answers may vary.

9 Cosine curve shifted to the right by $\frac{2\pi}{3}$.
$y = \cos\left(x - \frac{2\pi}{3}\right)$

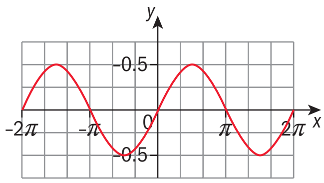
10 Sine curve shifted upwards by 1 unit. $y = \sin x + 1$

11 Tangent curve shifted right by $\frac{\pi}{4}$. $y = \tan\left(x - \frac{\pi}{4}\right)$

12 Cosine curve shifted right by $\frac{\pi}{4}$ and downwards by 1.5 units. $y = \cos\left(x - \frac{\pi}{4}\right) - 1.5$

Exercise 13J

1



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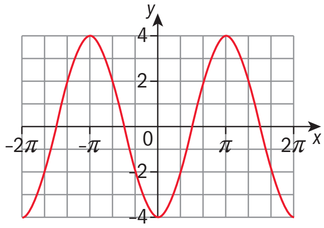

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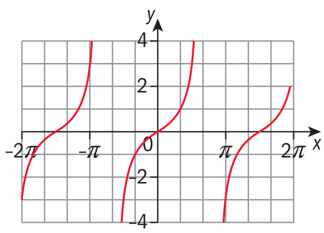

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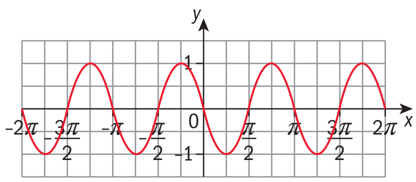

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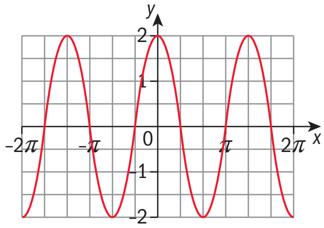

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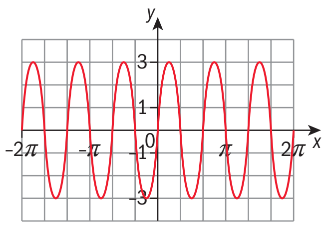

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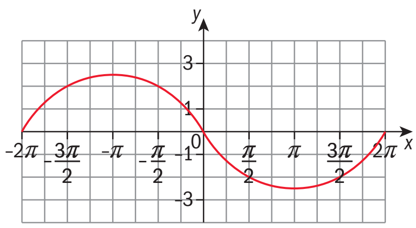

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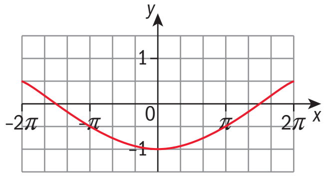


9 Sine graph, amplitude of functions is 7.5.
$y = 7.5 \sin x$

10 Cosine graph, period of functions is 8π.
$y = \cos(0.25x)$

11 Tangent graph, period of functions is 4π.
$y = \tan(0.25x)$

12 Cosine graph, reflected in x-axis, amplitude is 3, period is 4π. $y = -3\cos(0.25x)$

Exercise 13K

1 Want to write as $y = a\sin(b(x + c)) + d$ and $y = p\cos(q(x + r)) + s$

Amplitude: $a = p = \frac{2-(-5)}{2} = 3.5$

Period $= 2\pi$. So $b = q = \frac{2\pi}{2\pi} = 1$.

Vertical shift $= \frac{2+(-5)}{2} = -1.5 = d = 5$.

Horizontal shift: $c = \frac{-2\pi}{3}$, $r = \frac{5\pi}{6}$

So $y = 3.5\sin\left(x - \frac{2\pi}{3}\right) - 1.5$,

$y = 3.5\cos\left(x + \frac{5\pi}{6}\right) - 1.5$

2 Amplitude: $a = p = 1$

Period $= \frac{5\pi}{3} - \left(\frac{-7\pi}{3}\right) = \frac{12\pi}{3} = 4\pi$

So $b = q = \frac{2\pi}{4\pi} = \frac{1}{2}$

Vertical shift $= \frac{-3-1}{2} = -2 = d = s$

Horizontal shift: $c = \frac{4\pi}{3}$, $r = \frac{\pi}{3}$

So $y = \sin\left(\frac{1}{2}\left(x + \frac{4\pi}{3}\right)\right) - 2$,

$y = \cos\left(\frac{1}{2}\left(x + \frac{\pi}{3}\right)\right) - 2$

3 Amplitude: $a = p = \frac{3-(-1)}{2} = 2$.

Period $= \frac{\pi}{4} - \left(\frac{-3\pi}{4}\right) = \pi$.

So $b = q = \frac{2\pi}{\pi} = 2$.

Vertical shift: $d = s = \frac{3-1}{2} = 1$

Horizontal shift: $c = 0$, $r = \frac{-\pi}{4}$

So $y = 2\sin(2x) + 1$, $y = 2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$

4 Amplitude: $a = p = \frac{5-(-5)}{2} = 5$.

Period $= 2\pi - (-\pi) = 3\pi$.

So $b = q = \frac{2\pi}{3\pi} = \frac{2}{3}$

Vertical shift: $d = s = 0$

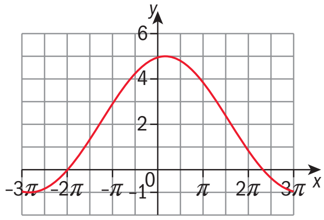
Horizontal shift: $c = \frac{-\pi}{2}$, $r = \frac{-5\pi}{4}$

So $y = 5\sin\left(\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right)$, $y = 5\cos\left(\frac{2}{3}\left(x - \frac{5\pi}{4}\right)\right)$

5 Amplitude = 3, period: $\frac{2\pi}{x} = \frac{1}{3}$,

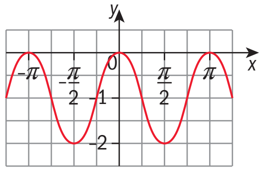
$x = \frac{2\pi}{\frac{1}{3}} = 6\pi$.

Vertical shift $= \frac{5-1}{2} = 2$, horizontal shift $= \frac{\pi}{6}$.



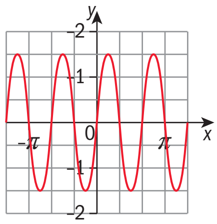
6 Amplitude = 1, period: $\frac{2\pi}{x} = -2$, $x = -\pi$.

Vertical shift $= \frac{0-2}{2} = -1$, horizontal shift $= -\frac{\pi}{4}$.



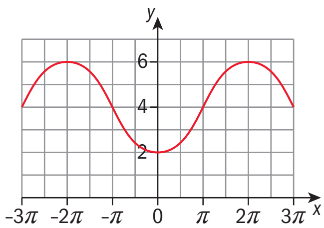
7 Amplitude = 1.5, period: $\frac{2\pi}{x} = 3$, $x = \frac{2\pi}{3}$.

Vertical shift = 0, horizontal shift $= -\frac{\pi}{2}$.



8 Amplitude = 2, period: $\frac{2\pi}{x} = \frac{1}{2}$, $x = 4\pi$.

Vertical shift = 4, horizontal shift = 0.



Exercise 13L

1 **a** period: $2 - 0 = 2$

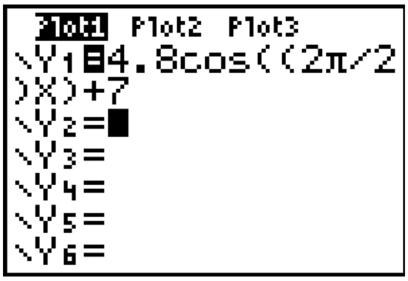
amplitude: $\frac{11.8 - 2.2}{2} = 4.8$

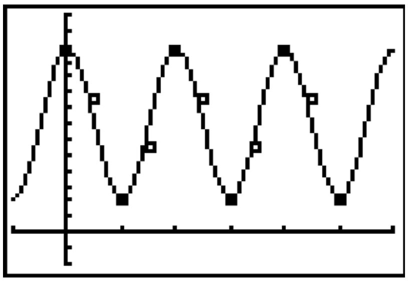
vertical shift: $\frac{11.8 + 2.2}{2} = 7$

horizontal shift: 0 (first maximum)

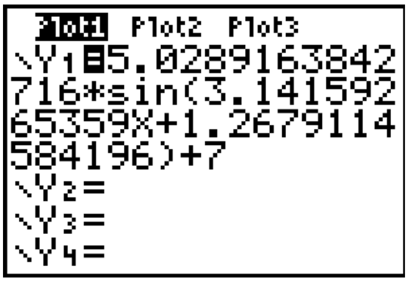
b $y = 4.8\cos\left(\frac{2\pi}{2}x\right) + 7$

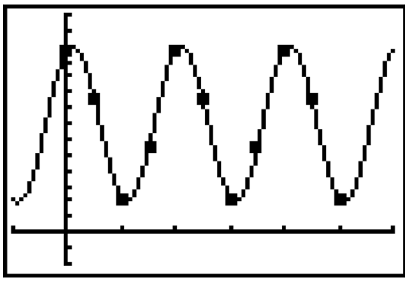
c





d





2 **a** period: $55 - 25 = 30$

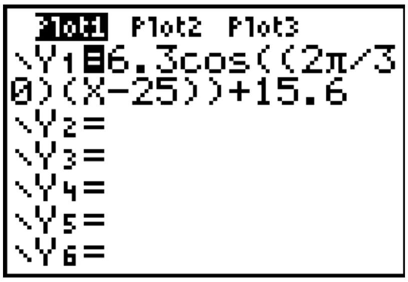
amplitude: $\frac{21.9 - 9.3}{2} = 6.3$

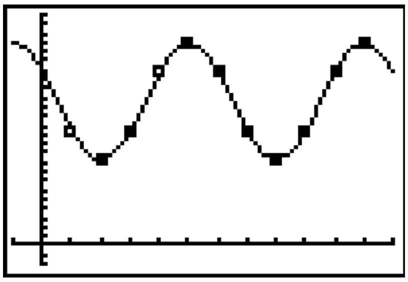
vertical shift: $\frac{21.9 + 9.3}{2} = 15.6$

horizontal shift: 25 (first maximum)

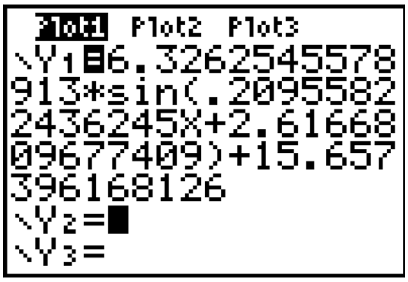
b $y = 6.3\cos\left(\frac{2\pi}{30}(x - 25)\right) + 15.6$

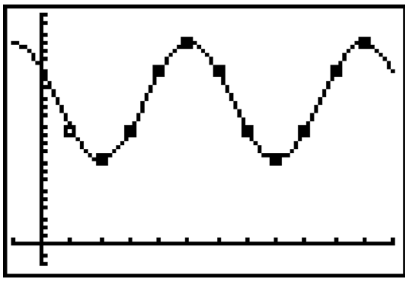
c





d





3 **a** period: $20 - 4 = 16$

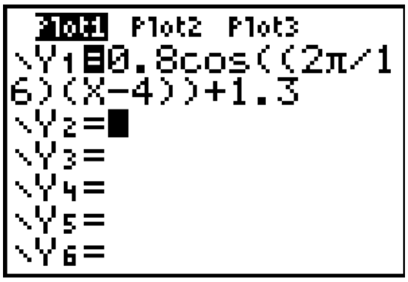
amplitude: $\frac{2.1 - 0.5}{2} = 0.8$

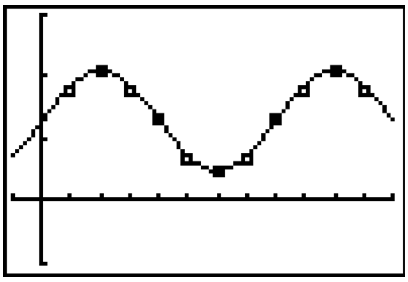
vertical shift: $\frac{2.1 + 0.5}{2} = 1.3$

horizontal shift: 4 (first maximum)

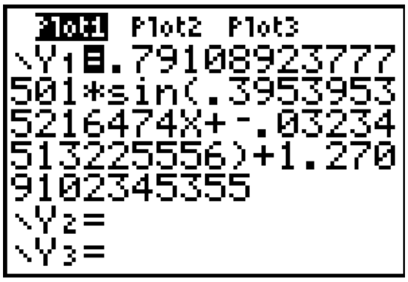
b $y = 0.8\cos\left(\frac{2\pi}{16}(x - 4)\right) + 1.3$

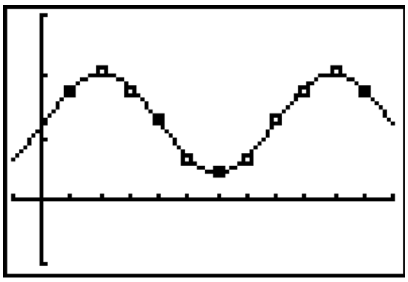
c





d





Exercise 13M

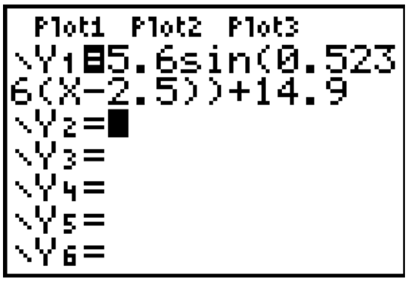
1 **a** period: $\frac{2\pi}{0.5236}$

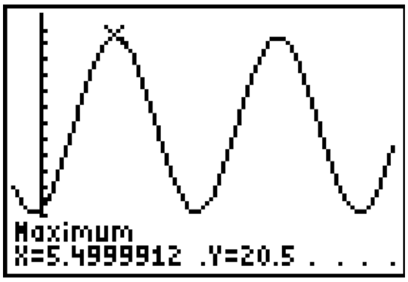
approximately 12 hours

b $d(0) = 5.6\sin(0.5236(0 - 2.5)) + 14.9 \approx 9.49\,\text{m}$

c $d(14) = 5.6\sin(0.5236(14 - 2.5))$
$+ 14.9 \approx 13.5\,\text{m}$

d

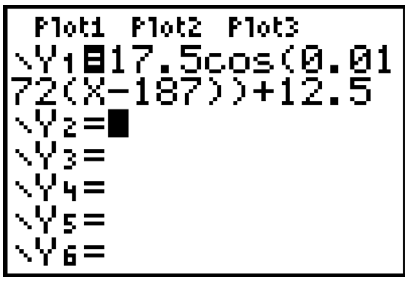


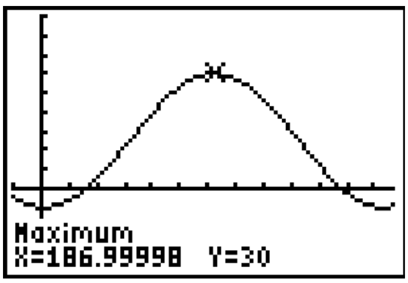


first maximum at about 05:30

2 **a** $T(32) = 17.5\cos(0.0172(32 - 187))$
$+ 12.5 \approx -3.06\,°\text{C}$

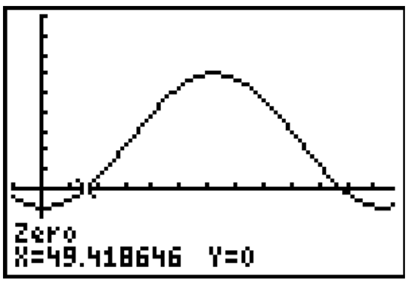
b high temp: $12.5 + 17.5 = 30\,°\text{C}$

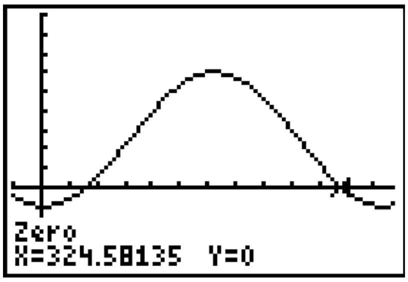




day 187 (about 6 July)

c





about 90 days

3 **a** after 10 minutes, the wheel will be at the maximum height, 46 m

b period: 20 min

amplitude: $\frac{46 - 1}{2} = 22.5$

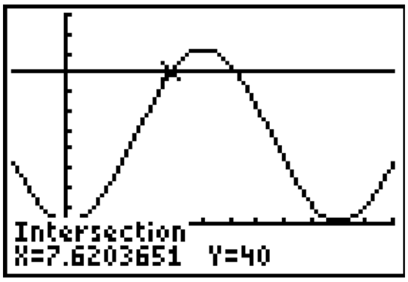
vertical shift: $\frac{46 + 1}{2} = 23.5$

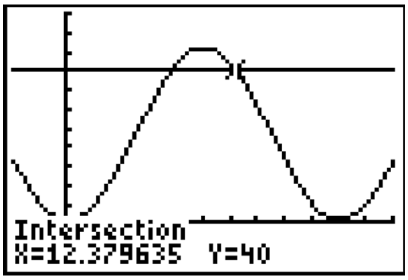
horizontal shift: 5 min

$h(t) = 22.5\sin\left(\frac{2\pi}{20}(t - 5)\right) + 23.5$

c $h(3) = 22.5\sin\left(\frac{2\pi}{20}(3 - 5)\right) + 23.5 \approx 10.3\text{ m}$

d





4.8 minutes

4 **a** period: 12

amplitude: $\frac{37 - 5}{2} = 16$

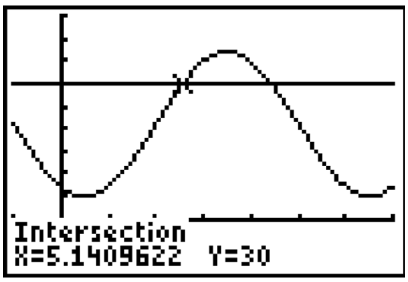
vertical shift: $\frac{37 + 5}{2} = 21$

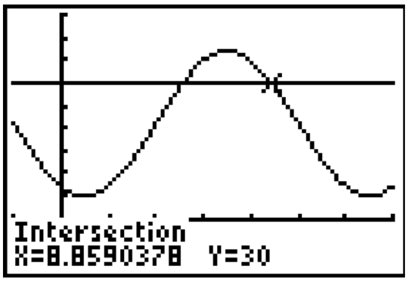
horizontal shift: 1 (first minimum) or 7 (first maximum)

$g(x) = -16\cos\left(\frac{2\pi}{12}(x - 1)\right) + 21$

b $g(x) = -16\cos\left(\frac{2\pi}{12}(4 - 1)\right) + 21 = 21\text{ gallons}$

c





early May and late August

Review exercise

1 **a** $\cos 110 = -\cos 70 = -0.342$

b $\cos 250 = -\cos 70 = -0.342$

c $\cos(-290) = \cos 70 = 0.342$

2 **a** $\sin 140 = \sin 40 = 0.643$

b $\sin 320 = -\sin 40 = -0.643$

c $\sin(-140) = -\sin 40 = -0.643$

3 **a** $\cos x = -\frac{1}{2}$

$x = \pm 120°, \pm 240°$

b $\tan x = \frac{1}{\sqrt{3}}$

$x = -330°, -150°, 30°, 210°$

c $2\sin^2 x - \sin x = 1$

$2\sin^2 x - \sin x - 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = 1$

$x = -150°, -30°, 210°, 330°,$

or $x = -270°, 90°$

4 $\sin 2x + \sin x = 0$

$2\sin x \cos x + \sin x = 0$

$\sin x(2\cos x + 1) = 0$

$\sin x = 0$ or $\cos x = -\frac{1}{2}$

$x = 0, \pi$ or $x = \frac{2\pi}{3}$

5 **a** **i** amplitude: $a = \frac{11-1}{2} = 5$

horizontal shift: $c = 4$

vertical shift: $d = \frac{11+1}{2} = 6$

ii $b = \frac{2\pi}{period}$, and the period is 8. $b = \frac{2\pi}{8} = \frac{\pi}{4}$

b $4 < x < 8$

6 **a** $\sin^2 x + \cos^2 x = 1$

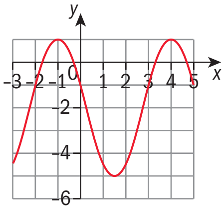
$\sin^2 x + \left(\frac{2}{5}\right)^2 = 1 \to \sin^2 x = 1 - \left(\frac{2}{5}\right)^2 = \frac{21}{25}$

$\sin x = \frac{\sqrt{21}}{5}$

b $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\left(\frac{\sqrt{21}}{5}\right)}{\left(\frac{2}{5}\right)} = \frac{\sqrt{21}}{2}$

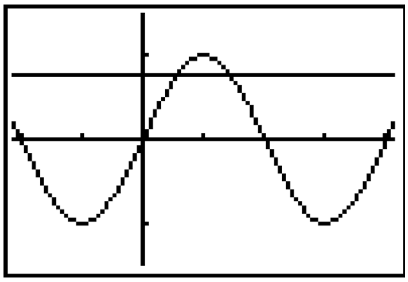
c $\sin(2x) = 2\sin x \cos x = 2\left(\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right) = \frac{4\sqrt{21}}{25}$

7



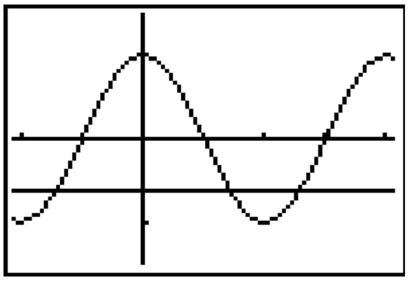
Review exercise

1 **a**



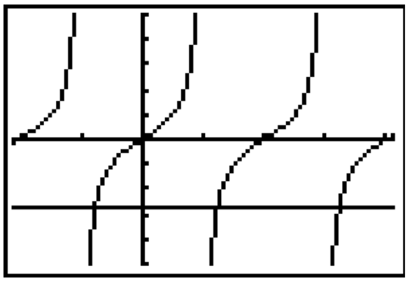
$48.6°, 131.4°$

b



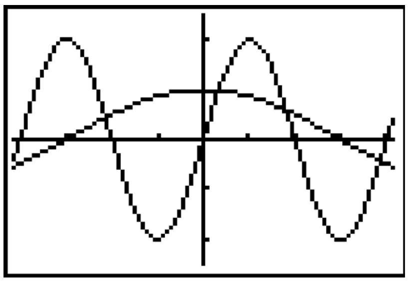
$\pm 129°, 231°$

c



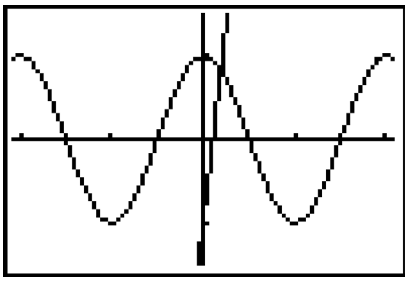
$-70.3°, 109.7°, 289.7°$

2 **a**



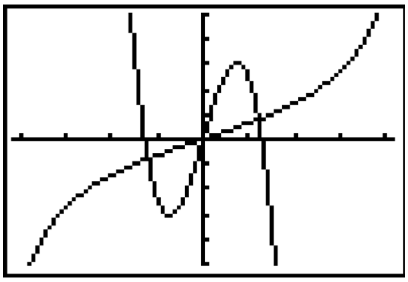
$-3.36, 0.515, 2.85, 6.06$

b



0.607

c



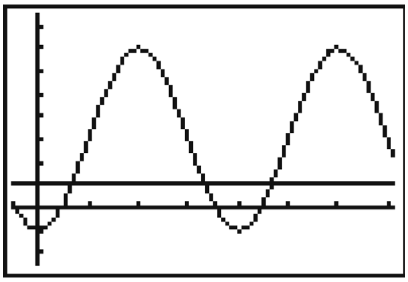
$\pm 1.89, 0$

3 **a** amplitude: $\frac{7-(-1)}{2} = 4 \rightarrow a = -4$

period: $4 - 0 = 4 \rightarrow b = \frac{2\pi}{period} = \frac{2\pi}{4} = \frac{\pi}{2}$

vertical shift: $\frac{7+(-1)}{2} = 3$

b $f(x) = -4\cos\left(\frac{\pi}{2}x\right) + 3 = 1$

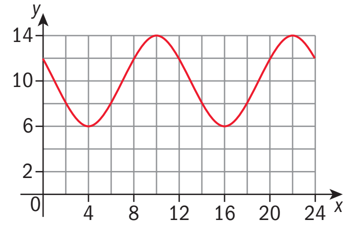


0.667, 3.33, 4.67

4 **a** amplitude: $P = \frac{14-6}{2} = 4$

horizontal shift: $Q = \frac{10+4}{2} = 7$

b $D(t) = 4\sin\left(\frac{\pi}{6}(t-7)\right) + 10$



c $t = 2$, at 2:00

d from 2:00–6:00, and again from 14:00–18:00

8 hours

5 **a** amplitude: $A = \frac{15-9.35}{2} = 2.825$

vertical shift: $B = \frac{15+9.35}{2} = 12.175$

b $h(x) = 2.825 \sin(0.0172(x - 86)) + 12.175$

$h(32) = 2.825 \sin(0.0172(32 - 86))$
$+ 12.175 \approx 9.91$ hours

14 Calculus with trigonometric functions

Answers

Skills check

1 **a** $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

b $\sin\frac{3\pi}{2} = -\sin\frac{\pi}{2} = -1$

c $\tan\frac{11\pi}{6} = \tan\frac{-\pi}{6} = -\tan\frac{\pi}{6} = \frac{-\sqrt{3}}{3}$

d $\sin\frac{4\pi}{3} = \sin\frac{-\pi}{3} = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}$

2 **a** $1 + \tan x = \sin^2 x + \cos^2 x$

$1 + \tan x = 1$

$\tan x = 0$

$\Rightarrow x = 0, \pi, 2\pi$

b $\sin 2x - \cos x = 0$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1 = 0)$

$\cos x = 0, 2\sin x = 1 \Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

c $\sin^2 x = 1 + \cos x$

$1 - \cos^2 x = 1 + \cos x$

$-\cos^2 x = \cos x$

$\cos x(1 + \cos x) = 0$

$\cos x = 0, \cos x = -1$

$\Rightarrow x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

3 **a** $f(x) = 2x^3e^x$

Use chain rule. $f'(x) = 3\times 2x^2e^x + 2x^3\times e^x$

$= 6x^2e^x + 2x^3e^x$

b $f(x) = x\ln(x^2)$

Use chain rule. To differentiate $\ln(x^2)$, use the subsitution $u = x^2$ to get $\frac{2}{x}$.

Then, $f'(x) = 1\times\ln(x^2) + x\times\frac{2}{x}$

$= \ln(x^2) + 2$

c $f(x) = \frac{x-5}{x^2+4}$

Use quotient rule. $f'(x) = \frac{1\times(x^2+4)-(x-5)\times 2x}{(x^2+4)^2}$

$= \frac{x^2+4-2x^2+10x}{(x^2+4)^2}$

$= \frac{-x^2+10x+4}{(x^2+4)^2}$

d $f(x) = \frac{\ln x}{x}$

Use quotient rule. $f'(x) = \frac{\frac{1}{x}\times x - \ln x\times 1}{x^2}$

$= \frac{1-\ln x}{x^2}$

Exercise 14A

1 $f(x) = 3\sin x - 2\cos x$

$f'(x) = 3\cos x - 2(-\sin x) = 3\cos x + 2\sin x$

2 $y = \tan(3x)$

$y' = \left[\frac{1}{\cos^2(3x)}\right](3) = \frac{3}{\cos^2 3x}$

3 $y = \frac{2}{\sin x} = 2(\sin x)^{-1}$

$y' = 2\left[-(\sin x)^{-2}(\cos x)\right] = -\frac{2\cos x}{\sin^2 x}$

4 $s(t) = \cos^2 t = (\cos t)^2$

$s'(t) = 2(\cos t)(-\sin t) = -2\sin t\cos t$ or $-\sin(2t)$

5 $f(x) = \sin\sqrt{x} = \sin(x)^{\frac{1}{2}}$

$f'(x) = \left[\cos(x)^{\frac{1}{2}}\right]\left[\frac{1}{2}x^{-\frac{1}{2}}\right] = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

6 $y = \tan^2 x = (\tan x)^2$

$y' = 2(\tan x)\left(\frac{1}{\cos^2 x}\right) = \frac{2\tan x}{\cos^2 x}$ or $\frac{2\sin x}{\cos^3 x}$

7 $y = \cos\frac{x}{2} + \sin(4x) = \cos\left(\frac{1}{2}x\right) + \sin(4x)$

$y' = \left[-\sin\left(\frac{1}{2}x\right)\right]\left(\frac{1}{2}\right) + \left[\cos(4x)\right](4)$

$= -\frac{1}{2}\sin\frac{x}{2} + 4\cos(4x)$

8 $f(x) = \frac{1}{\cos(2x)} = \left[\cos(2x)\right]^{-1}$

$f'(x) = -1\left[\cos(2x)\right]^{-2}\left[(-\sin(2x))(2)\right] = \frac{2\sin(2x)}{\cos^2(2x)}$

9 $y = \frac{4}{\sin^2(\pi x)} = 4\left[\sin(\pi x)\right]^{-2}$

$y' = 4\left[-2\left[\sin(\pi x)\right]^{-3}\left[\cos(\pi x)\right](\pi)\right] = \frac{-8\pi\cos(\pi x)}{\sin^3(\pi x)}$

10 $f(x) = \sin(\sin x)$

$f'(x) = \left[\cos(\sin x)\right](\cos x)$

11 **a** $\frac{d}{dx}\left[\tan\left(x^3\right)\right] = \frac{1}{\cos^2\left(x^3\right)}(3x^2) = \frac{3x^2}{\cos^2\left(x^3\right)}$

b $\frac{d}{dx}\left[\cos^4 x\right] = \frac{d}{dx}\left[(\cos x)^4\right] = \left[4(\cos x)^3\right](-\sin x)$

$= -4\cos^3 x\sin x$

12 **a** $y = \sin(3x-4)$;

$\frac{dy}{dx} = \left[\cos(3x-4)\right](3) = 3\cos(3x-4)$

b $\frac{d^2y}{dx^2} = 3\left[-\sin(3x-4)\right](3) = -9\sin(3x-4)$

Exercise 14B

1 $f(x) = \sin x - \cos x;\ x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$

$f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

$m_{\text{tangent}} = f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$

$m_{\text{normal}} = -1$

tangent line: $y - 1 = 1\left(x - \frac{\pi}{2}\right)$

normal line: $y - 1 = -1\left(x - \frac{\pi}{2}\right)$

2 $f(x) = 2\tan x;\ x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right) = 2(1) = 2$

$f'(x) = \frac{2}{\cos^2 x}$

$m_{\text{tangent}} = f'\left(\frac{\pi}{4}\right) = \frac{2}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{2}{\frac{1}{2}} = 4$

$m_{\text{normal}} = -\frac{1}{4}$

Tangent line: $y - 2 = 4\left(x - \frac{\pi}{4}\right)$

Normal line: $y - 2 = -\frac{1}{4}\left(x - \frac{\pi}{4}\right)$

3 $P\left(\frac{\pi}{2}, 0\right);\ y = \sin(2x)$

$y' = [\cos(2x)](2) = 2\cos(2x)$

$m = 2\cos\left(2\left(\frac{\pi}{2}\right)\right) = 2\cos\pi = 2(-1) = -2$

4 **a** $f(x) = \cos(2x)$

$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

b $f'(x) = [-\sin(2x)](2) = -2\sin(2x)$

c $m = f'\left(\frac{\pi}{3}\right) = -2\sin\left(\frac{2\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$

$y + \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{3}\right)$

5 $f(x) = 3\sin x$ for $0 \le x \le 2\pi$

tangent lines parallel to the line $y = \frac{3}{2}x + 4 \Rightarrow m = \frac{3}{2}$

$m = f'(x) = 3\cos x$

$3\cos x = \frac{3}{2}$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Exercise 14C

1 $f(x) = 6\cos\left(2x - \frac{\pi}{3}\right) + 3x$

$f'(x) = 6\left[-\sin\left(2x - \frac{\pi}{3}\right)\right](2) + 3 = -12\sin\left(2x - \frac{\pi}{3}\right) + 3$

2 $y = \frac{\sin x}{1 + \cos x}$

$y' = \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{(1+\cos x)}$

3 $f(x) = xe^x - e^x$

$f'(x) = (x)(e^x) + (e^x)(1) - e^x = xe^x$

4 $s(t) = \frac{1}{2}e^{\sin 2t}$

$s'(t) = \frac{1}{2}\left[e^{\sin 2t}(\cos 2t)(2)\right] = e^{\sin 2t}\cos 2t$

5 $f(x) = e^x(\sin x - \cos x)$

$f'(x) = (e^x)(\cos x - (-\sin x)) + (\sin x - \cos x)(e^x)$

$= e^x\cos x + e^x\sin x + e^x\sin x - e^x\cos x = 2e^x\sin x$

6 $s(t) = t\tan t$

$s'(t) = (t)\left(\frac{1}{\cos^2 t}\right) + (\tan t)(1) = \frac{t}{\cos^2 t} + \tan t$

7 $y = e^{3x}\cos 4x$

$y' = (e^{3x})\left[(-\sin 4x)(4)\right] + (\cos 4x)\left[(e^{3x})(3)\right]$

$= 3e^{3x}\cos 4x - 4e^{3x}\sin 4x$

8 $y = \sqrt{\tan 2x} = (\tan 2x)^{\frac{1}{2}}$

$y' = \frac{1}{2}(\tan 2x)^{\frac{-1}{2}}\left(\frac{1}{\cos^2(2x)}(2)\right) = \frac{1}{\cos^2(2x)\sqrt{\tan 2x}}$

9 $f(x) = (\ln x)(\cos x)$

$f'(x) = (\ln x)(-\sin x) + (\cos x)\left(\frac{1}{x}\right) = \frac{\cos x}{x} - \ln x\sin x$

10 $f(x) = \ln(\cos x)$

$f'(x) = \left(\frac{1}{\cos x}\right)(-\sin x) = -\frac{\sin x}{\cos x}$ or $-\tan x$

11 **a** $f(x) = \ln(3x^2)$

$f'(x) = \left(\frac{1}{3x^2}\right)(6x) = \frac{6x}{3x^2} = \frac{2}{x}$

b $g(x) = \sin\frac{x}{2} = \sin\left(\frac{1}{2}x\right)$

$g'(x) = \left[\cos\left(\frac{1}{2}x\right)\right]\left(\frac{1}{2}\right) = \frac{1}{2}\cos\frac{x}{2}$

c $h(x) = \ln(3x^2)\sin\frac{x}{2}$

$h'(x) = \left[\ln(3x^2)\right]\left[\frac{1}{2}\cos\frac{x}{2}\right] + \left(\sin\frac{x}{2}\right)\left(\frac{2}{x}\right)$

$= \frac{1}{2}\ln(3x^2)\cos\frac{x}{2} + \frac{2}{x}\sin\frac{x}{2}$

12 $f(x) = \frac{\sin x}{1 + \cos^2 x}$ and $f'(x) = \frac{\cos x(1 + a\cos^2 x + b\sin^2 x)}{(1+\cos^2 x)^2}$

$f(x) = \frac{\sin x}{1 + \cos^2 x}$

$f'(x) = \frac{(1+\cos^2 x)(\cos x) - (\sin x)\left[(2\cos x)(-\sin x)\right]}{(1+\cos^2 x)^2}$

$= \frac{\cos x\left[(1+\cos^2 x) + 2\sin^2 x\right]}{(1+\cos^2 x)^2} = \frac{\cos x\left(1 + \cos^2 x + 2\sin^2 x\right)}{(1+\cos^2 x)^2}$

$a = 1;\ b = 2$

Exercise 14D

1 $f(x)=\sqrt{3}\sin x+\cos x,\ 0\le x\le 2\pi$

$f'(x)=\sqrt{3}\cos x-\sin x$

$\sqrt{3}\cos x-\sin x=0\Rightarrow\sqrt{3}\cos x=\sin x$

$\Rightarrow\frac{\sin x}{\cos x}=\sqrt{3}$

$\Rightarrow\tan x=\sqrt{3}$

$\Rightarrow x=\frac{\pi}{3},\frac{4\pi}{3}$

sign of f': $+$ | $-$ | $+$

x: 0, $\frac{\pi}{3}$, $\frac{4\pi}{3}$, 2π

$f\left(\frac{\pi}{3}\right)=\sqrt{3}\sin\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)$

$=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}=\frac{3}{2}+\frac{1}{2}=2$

$f\left(\frac{4\pi}{3}\right)=\sqrt{3}\sin\left(\frac{4\pi}{3}\right)+\cos\left(\frac{4\pi}{3}\right)$

$=\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)-\frac{1}{2}=-\frac{3}{2}-\frac{1}{2}=-2$

relative minimum: $\left(\frac{4\pi}{3},-2\right)$;

relative maximum: $\left(\frac{\pi}{3},2\right)$

2 $f(x)=2\sin x+\cos 2x,\ 0\le x\le 2\pi$

$f'(x)=2\cos x+(-\sin 2x)(2)=2\cos x-2\sin 2x$

$2\cos x-2\sin 2x=0\Rightarrow 2\cos x-2(2\sin x\cos x)=0$

$\Rightarrow 2\cos x(1-2\sin x)=0$

$\Rightarrow 2\cos x=0$ or $1-2\sin x=0$

$\Rightarrow\cos x=0$ or $\sin x=\frac{1}{2}$

$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$

sign of f': $+$ | $-$ | $+$ | $-$ | $+$

x: 0, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{3\pi}{8}$, 2π

$f\left(\frac{\pi}{6}\right)=2\sin\left(\frac{\pi}{6}\right)+\cos\left(2\left(\frac{\pi}{6}\right)\right)=2\left(\frac{1}{2}\right)+\frac{1}{2}=\frac{3}{2}$

$f\left(\frac{5\pi}{6}\right)=2\sin\left(\frac{5\pi}{6}\right)+\cos\left(2\left(\frac{5\pi}{6}\right)\right)=2\left(\frac{1}{2}\right)+\frac{1}{2}=\frac{3}{2}$

$f\left(\frac{\pi}{2}\right)=2\sin\left(\frac{\pi}{2}\right)+\cos\left(2\left(\frac{\pi}{2}\right)\right)=2(1)-1=1$

$f\left(\frac{3\pi}{2}\right)=2\sin\left(\frac{3\pi}{2}\right)+\cos\left(2\left(\frac{3\pi}{2}\right)\right)=2(-1)-1=-3$

relative minimums: $\left(\frac{\pi}{2},1\right),\left(\frac{3\pi}{2},-3\right)$;

relative maximums: $\left(\frac{\pi}{6},\frac{3}{2}\right),\left(\frac{5\pi}{6},\frac{3}{2}\right)$

3 $f(x)=\sqrt{\sin x}=(\sin x)^{\frac{1}{2}},0\le x\le\pi$

$f'(x)=\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)=\frac{\cos x}{2\sqrt{\sin x}}$

$f'(x)=0\Rightarrow\cos x=0\Rightarrow x=\frac{\pi}{2}$

$f'(x)$ is undefined when

$2\sqrt{\sin x}=0\Rightarrow\sin x=0\Rightarrow x=0,\pi$

sign of f': $+$ | $-$

x: 0, $\frac{\pi}{2}$, π

$f'(x)=\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$

$f''(x)=\frac{1}{2}(\sin x)^{-\frac{1}{2}}(-\sin x)+(\cos x)\left[-\frac{1}{4}(\sin x)^{-\frac{3}{2}}(\cos x)\right]$

$=-\frac{1}{2}(\sin x)^{\frac{1}{2}}-\frac{1}{4}(\sin x)^{-\frac{3}{2}}(\cos^2 x)$

$=-\frac{1}{4}(\sin x)^{-\frac{3}{2}}\left[2\sin^2 x+\cos^2 x\right]=-\frac{2\sin^2 x+\cos^2 x}{4(\sin x)^{\frac{3}{2}}}$

$f''(x)=0\Rightarrow 2\sin^2 x+\cos^2 x=0$

$\Rightarrow 2\sin^2 x+(1-\sin^2 x)=0$

$\Rightarrow\sin^2 x+1=0$, which has no solutions.

sign of f': $-$ | $-$

x: 0, π

$f''(x)$ is undefined when $(\sin x)^{\frac{3}{2}}=0\Rightarrow x=0,\pi$

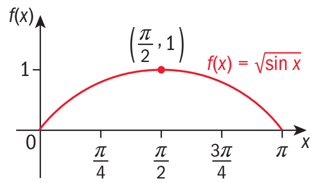
Decreasing when $f'(x)<0$: $\frac{\pi}{2}<x<\pi$

Increasing when $f'(x)>0$: $0<x<\frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right)=\sqrt{\sin\frac{\pi}{2}}=1\Rightarrow$ relative maximum: $\left(\frac{\pi}{2},1\right)$

Concave down when $f''(x)<0$: $0<x<\pi$

$\sqrt{\sin x}=0\Rightarrow x=0,\pi\Rightarrow$ x-intercepts are $(0,0)$ and $(\pi,0)$



4 $f(x)=\cos^2(2x)=\left[\cos(2x)\right]^2,0\le x\le\pi$

$f'(x)=2\left[\cos(2x)\right]\left[(-\sin(2x))(2)\right]$

$=-4\sin 2x\cos 2x$ or $-2\sin 4x$

$-2\sin 4x=0\Rightarrow 4x=0,\pi,2\pi,3\pi,4\pi$

$\Rightarrow x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi$

decreasing: $0<x<\frac{\pi}{4},\ \frac{\pi}{2}<x<\frac{3\pi}{4}$

increasing: $\frac{\pi}{4}<x<\frac{\pi}{2},\ \frac{3\pi}{4}<x<\pi$

sign of f': $-$ | $+$ | $-$ | $+$

x: 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π

$f\left(\frac{\pi}{4}\right)=\cos^2\left(\frac{\pi}{2}\right)=0$;

$f\left(\frac{\pi}{2}\right)=\cos^2(\pi)=1$;

$f\left(\frac{3\pi}{4}\right)=\cos^2\left(\frac{3\pi}{2}\right)=0$

relative minimum points: $\left(\frac{\pi}{4},0\right),\left(\frac{3\pi}{4},0\right)$

relative maximum point: $\left(\frac{\pi}{2},1\right)$

$f'(x) = -2\sin 4x \Rightarrow f''(x) = -2(\cos 4x)(4)$
$= -8\cos(4x)$

$-8\cos(4x) = 0 \Rightarrow 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

sign of f': – + – + –

x: 0, $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, $\frac{7\pi}{8}$, π

concave down: $\left(0, \frac{\pi}{8}\right), \left(\frac{3\pi}{8}, \frac{5\pi}{8}\right), \left(\frac{7\pi}{8}, \pi\right)$

concave up: $\left(\frac{\pi}{8}, \frac{3\pi}{8}\right), \left(\frac{5\pi}{8}, \frac{7\pi}{8}\right)$

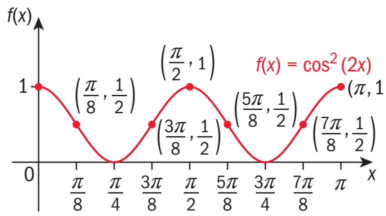
$f\left(\frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2};$

$f\left(\frac{3\pi}{8}\right) = \cos^2\left(\frac{3\pi}{4}\right) = \frac{1}{2};$

$f\left(\frac{5\pi}{8}\right) = \cos^2\left(\frac{5\pi}{4}\right) = \frac{1}{2};$

$f\left(\frac{7\pi}{8}\right) = \cos^2\left(\frac{7\pi}{4}\right) = \frac{1}{2}$

inflexion points: $\left(\frac{\pi}{8}, \frac{1}{2}\right), \left(\frac{3\pi}{8}, \frac{1}{2}\right), \left(\frac{5\pi}{8}, \frac{1}{2}\right), \left(\frac{7\pi}{8}, \frac{1}{2}\right)$



5 **a** $f(x) = \cos 2x + \cos^2 x = \cos 2x + (\cos x)^2$

$f'(x) = (-\sin 2x)(2) + 2(\cos x)(-\sin x)$

$= -2\sin 2x - 2\sin x\cos x = -2\sin 2x - \sin 2x$

$= -3\sin 2x$

b $f'(x) = -3\sin 2x = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = 0, \pi, 2\pi$

$\Rightarrow x = 0, \frac{\pi}{2}, \pi$ on the interval $0 \le x \le \pi$

sign of f': – +

x: 0, $\frac{\pi}{2}$, π

$f\left(\frac{\pi}{2}\right) = \cos\left(2\left(\frac{\pi}{2}\right)\right) + \cos^2\left(\frac{\pi}{2}\right) = -1 + 0 = -1$

relative minimum point: $\left(\frac{\pi}{2}, -1\right)$

c $f'(x) = -3\sin 2x$

$f''(x) = -3[(\cos 2x)(2)] = -6\cos 2x$

d $-6\cos 2x = 0 \Rightarrow \cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

on the interval $0 \le x \le \pi$

sign of f': – + –

x: 0, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, π

$f\left(\frac{\pi}{4}\right) = \cos\left(2\left(\frac{\pi}{4}\right)\right) + \cos^2\left(\frac{\pi}{4}\right) = 0 + \frac{1}{2} = \frac{1}{2}$

$f\left(\frac{3\pi}{4}\right) = \cos\left(2\left(\frac{3\pi}{4}\right)\right) + \cos^2\left(\frac{3\pi}{4}\right) = 0 + \frac{1}{2} = \frac{1}{2}$

inflexion points: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{3\pi}{4}, \frac{1}{2}\right)$

6 **a** **i** $f(x) = \pi + x\sin x$

$f'(x) = 0 + [(x)(\cos x) + (\sin x)(1)]$

$= x\cos x + \sin x$

ii $f''(x) = ax\sin x + b\cos x$

$f''(x) = [(x)(-\sin x) + (\cos x)(1)] + \cos x$

$= -x\sin x + 2\cos x \Rightarrow a = -1$ and $b = 2$

b **i** $f'(x) = 0$ for $0 \le x \le 2\pi$

Use a GDC to solve: $x\cos x + \sin x = 0$

$\Rightarrow x \approx 2.03, 4.91$

ii $f''(2.03) \approx -2.71 < 0 \Rightarrow$ relative maximum at $x = 2.03$

$f''(4.91) \approx 5.21 > 0 \Rightarrow$ relative minimum at $x = 4.91$

7 **a** $f(x) = x^2\cos x$

$f'(x) = (x^2)(-\sin x) + (\cos x)(2x)$

$= -x^2\sin x + 2x\cos x$

b $f'(x) = -x^2\sin x + 2x\cos x = 0 \Rightarrow$

$x \approx 1.077, 3.644$ on the interval $0 \le x \le 5$

$f(0) = 1$

$f(1.077) \approx 0.550$

$f(3.644) \approx -11.6$

$f(5) \approx 7.09$

minimum: –11.6 maximum: 7.09

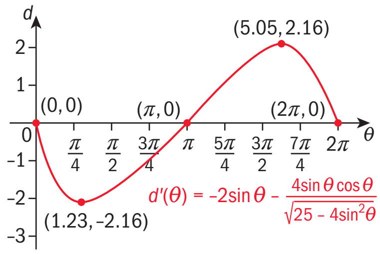
8 **a** $d(\theta) = 2\cos\theta + \sqrt{25 - 4\sin^2\theta}$

$= 2\cos\theta + (25 - 4\sin^2\theta)^{\frac{1}{2}}$

$d'(\theta) = -2\sin\theta + \frac{1}{2}(25 - 4\sin^2\theta)^{-\frac{1}{2}}(-8\sin\theta\cos\theta)$

$= -2\sin\theta - \frac{2\sin 2\theta}{\sqrt{25 - 4\sin^2\theta}}$

b



c **i** The blade is closest to the center of the wheel when $d(\theta)$ has a relative minimum or at an endpoint. There is a relative minimum when $d'(\theta)$ changes from negative to positive at $\theta = \pi$. Testing the endpoints and critical numbers we find $d(0) = 7$, $d(2\pi) = 7$ and $d(\pi) = 3$. So the closest distance is 3 meters and it occurs when the angle of rotation is π.

ii The distance is changing fastest when $d'(\theta)$ has a relative minimum or maximum. This occurs when θ is 1.23 radians or 5.05 radians.

Exercise 14E

1 $\int(2\cos x+3\sin x)dx=2\int\cos x dx+3\int\sin x dx$
$=2(\sin x)+3(-\cos x)+C$
$=2\sin x-3\cos x+C$

2 $\int\left(x^2+\cos\left(\frac{1}{3}x\right)\right)dx=\int x^2dx+\int\cos\left(\frac{1}{3}x\right)dx$
$=\frac{1}{3}x^3+3\sin\left(\frac{1}{3}x\right)+C$

3 $\int\pi\sin(\pi x)dx=\pi\int\sin(\pi x)dx$
$=\pi\left[\frac{1}{\pi}(-\cos(\pi x))\right]+C$
$=-\cos(\pi x)+C$

4 $\int\sin(2x+3)dx=\frac{1}{2}\left[-\cos(2x+3)\right]+C$
$=-\frac{1}{2}\cos(2x+3)+C$

5 $\int 20x^3\cos(5x^4)dx\Rightarrow u=5x^4;\ \frac{du}{dx}=20x^3$
$\int 20x^3\cos(5x^4)dx=\int\left(\frac{du}{dx}\right)\cos(u)dx$
$=\int\cos u du=\sin u+C$
$=\sin(5x^4)+C$

6 $\int(2x-1)\cos(4x^2-4x)dx\Rightarrow u=4x^2-4x;$
$\frac{du}{dx}=8x-4\Rightarrow\frac{du}{dx}=4(2x-1)\Rightarrow\frac{1}{4}\left(\frac{du}{dx}\right)=2x-1$
$\int(2x-1)\cos(4x^2-4x)dx=\int\frac{1}{4}\left(\frac{du}{dx}\right)\cos(u)dx$
$=\frac{1}{4}\int\cos u du=\frac{1}{4}\sin u+C$
$=\frac{1}{4}\sin(4x^2-4x)+C$

7 $\int\frac{e^{\tan(3x)}}{\cos^2(3x)}dx=\int e^{\tan(3x)}\left(\frac{1}{\cos^2(3x)}\right)dx\Rightarrow u=\tan(3x);$
$\frac{du}{dx}=\frac{1}{\cos^2(3x)}(3)\Rightarrow\frac{1}{3}\left(\frac{du}{dx}\right)=\frac{1}{\cos^2(3x)}$
$\int\frac{e^{\tan(3x)}}{\cos^2(3x)}dx=\int e^{\tan(3x)}\left(\frac{1}{\cos^2(3x)}\right)dx=\int e^u\left[\frac{1}{3}\left(\frac{du}{dx}\right)\right]dx$
$=\frac{1}{3}\int e^u du=\frac{1}{3}e^u+C=\frac{1}{3}e^{\tan(3x)}+C$

8 $\int\frac{\cos(\ln x)}{x}dx=\int\left[\cos(\ln x)\right]\left(\frac{1}{x}\right)dx\Rightarrow u=\ln x;\ \frac{du}{dx}=\frac{1}{x}$
$\int\frac{\cos(\ln x)}{x}dx=\int\left[\cos(\ln x)\right]\left(\frac{1}{x}\right)dx$
$=\int\cos u\left(\frac{du}{dx}\right)dx$
$=\int\cos u du=\sin u+C=\sin(\ln x)+C$

9 $\int\cos x\sin^2 x dx=\int\cos x(\sin x)^2 dx$
$\Rightarrow u=\sin x;\ \frac{du}{dx}=\cos x$
$\int\cos x\sin^2 x dx=\int\cos x(\sin x)^2 dx=\int\left(\frac{du}{dx}\right)u^2dx$
$=\int u^2du=\frac{1}{3}u^3+C=\frac{1}{3}\sin^3 x+C$

10 $\int\frac{\sin x}{\cos x}dx$, for $\cos x>0$
$\int\frac{\sin x}{\cos x}dx=\int\frac{1}{\cos x}(\sin x)dx\Rightarrow u=\cos x;\ \frac{du}{dx}=-\sin x$
$\Rightarrow-\frac{du}{dx}=\sin x$
$\int\frac{\sin x}{\cos x}dx=\int\frac{1}{\cos x}(\sin x)dx=\int\frac{1}{u}\left(-\frac{du}{dx}\right)dx$
$=-\int\frac{1}{u}du=-\ln u+C=-\ln(\cos x)+C,\ \cos x>0$

11 a $f(x)=e^{\sin x}\cos x$
$f'(x)=\left(e^{\sin x}\right)(-\sin x)+(\cos x)\left[\left(e^{\sin x}\right)(\cos x)\right]$
$=-e^{\sin x}\sin x+e^{\sin x}\cos^2 x$

b $\int e^{\sin x}\cos x dx\Rightarrow u=\sin x;\ \frac{du}{dx}=\cos x$
$\int e^u\left(\frac{du}{dx}\right)dx=\int e^u du=e^u+C=e^{\sin x}+C$

12 a $f(x)=\ln(\cos x)$
$f'(x)=\frac{1}{\cos x}(-\sin x)=-\frac{\sin x}{\cos x}=-\tan x$

b $\int\tan x\ln(\cos x)dx\Rightarrow u=\ln(\cos x);\ \frac{du}{dx}=-\tan x$
$\Rightarrow-\frac{du}{dx}=\tan x$
$\int\left(-\frac{du}{dx}\right)u dx=-\int u du=-\frac{1}{2}u^2+C=-\frac{1}{2}\left[\ln(\cos x)\right]^2+C$

Exercise 14F

1 $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos x dx=\left[\sin x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}=\sin\left(\frac{\pi}{3}\right)-\sin\left(-\frac{\pi}{3}\right)$
$=\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)=\sqrt{3}$
$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos x dx\approx 1.73$ and $\sqrt{3}\approx 1.73$

2 $\int_0^{\pi}(2\sin x+\sin 2x)dx=\left[-2\cos x-\frac{1}{2}\cos 2x\right]_0^{\pi}$
$=\left[-2\cos(\pi)-\frac{1}{2}\cos(2\pi)\right]$
$-\left[-2\cos(0)-\frac{1}{2}\cos(0)\right]$
$=\frac{3}{2}+\frac{5}{2}=4$
$\int_0^{\pi}(2\sin x+\sin 2x)dx=4$

3 $\int_0^{\frac{\pi}{2}}\cos\left(\frac{2}{3}x\right)dx=\left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_0^{\frac{\pi}{2}}$
$=\left[\frac{3}{2}\sin\left(\frac{2}{3}\left(\frac{\pi}{2}\right)\right)\right]-\left[\frac{3}{2}\sin\left(\frac{2}{3}(0)\right)\right]$
$=\frac{3\sqrt{3}}{4}$
$\int_0^{\frac{\pi}{2}}\cos\left(\frac{2}{3}x\right)dx\approx 1.30$ and $\frac{3\sqrt{3}}{4}\approx 1.30$

4 $\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{3}} e^x\cos(e^x)\,dx \Rightarrow u = e^x; \frac{du}{dx} = e^x;$

when $x = \ln\frac{\pi}{4}$ then $u = \frac{\pi}{4}$ and

when $x = \ln\frac{\pi}{3}$ then $u = \frac{\pi}{3}$

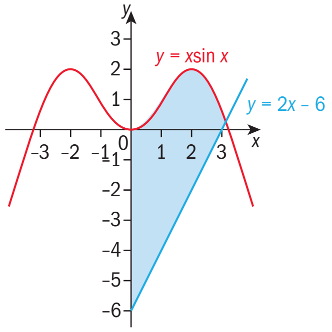
$$\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{3}} e^x\cos(e^x)\,dx = \int_{x=\ln\frac{\pi}{4}}^{x=\ln\frac{\pi}{3}} \left(\frac{du}{dx}\right)\cos(u)\,dx = \int_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} \cos u\,du$$

$$= \left[\sin u\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}-\sqrt{2}}{2}$$

$$\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{3}} e^x\cos(e^x)\,dx \approx 0.159 \text{ and } \frac{\sqrt{3}-\sqrt{2}}{2} \approx 0.159$$

Exercise 14G

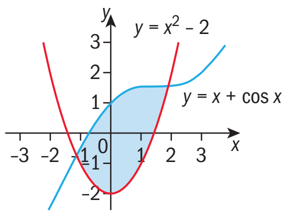
1 $y = x\sin x$ and $2x - 6$



$x\sin x = 2x - 6 \Rightarrow x = 3.1$

$$\int_0^{3.041} (x\sin x - 2x + 6)\,dx \approx 12.1$$

2 $y = x^2 - 2$ and $y = x + \cos x$



$x^2 - 2 = x + \cos x \Rightarrow x \approx -1.135, 1.891$

$$\int_{-1.135}^{1.891} \left[(x+\cos x) - (x^2-2)\right] \approx 6.31$$

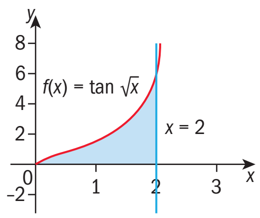
3 $\int_0^k \cos x\,dx = \frac{1}{2}$ and $0 \le k \le \frac{\pi}{2}$

$$\int_0^k \cos x\,dx = \frac{1}{2} \Rightarrow \left[\sin x\right]_0^k = \frac{1}{2}$$

$$\Rightarrow \sin k - \sin 0 = \frac{1}{2} \Rightarrow \sin k = \frac{1}{2}$$

$$\Rightarrow k = \frac{\pi}{6}$$

4 **a**



$$\int_0^2 \tan\sqrt{x}\,dx \approx 3.97$$

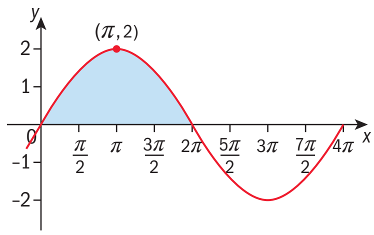
b $\pi\int_0^2 \left(\tan\sqrt{x}\right)^2 dx \approx 38.3$

5 **a** $f(x) = a\sin(bx)$

The sine function has a vertical stretch by a factor of $2 \Rightarrow a = 2$.

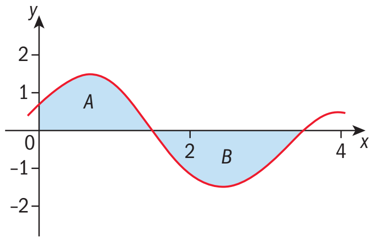
Since the period of f is 4π we have

$$\frac{2\pi}{|b|} = 4\pi \Rightarrow b = \frac{1}{2}$$



$$\int_0^{2\pi} 2\sin\left(\frac{1}{2}x\right)dx = 8$$

6 $y = \cos x + \sin 2x$



a **i** $y = \cos x + \sin 2x \Rightarrow y = \cos x + 2\sin x\cos x$

$\Rightarrow y = \cos x(1 + 2\sin x)$

$y = \cos x(c + d\sin x) \Rightarrow c = 1$ and $d = 2$

ii $\cos x(1 + 2\sin x) = 0 \Rightarrow \cos x = 0$ or $\sin x = -\frac{1}{2}$

$$\Rightarrow x = \frac{\pi}{2}, \frac{7\pi}{6}$$

b **i** $\int_0^{\frac{\pi}{2}} \left[\cos x + \sin(2x)\right]dx = 2$

ii $2 - \int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} \left[\cos x + \sin(2x)\right]dx = 4.25$

c $\pi\int_0^{\frac{\pi}{2}} \left[\cos x + \sin(2x)\right]^2 dx \approx 9.12$

Exercise 14H

1 **a** $s(t) = e^t\sin t$

$v(t) = s'(t) = (e^t)(\cos t) + (\sin t)(e^t)$

$= e^t(\cos t + \sin t)$

b $v(t) = e^t(\cos t + \sin t)$

$a(t) = v'(t) = (e^t)(-\sin t + \cos t) + (\cos t + \sin t)(e^t)$

$= 2e^t\cos t$

2 **a** $s(t) = 1 - 2\sin t$

$v(t) = s'(t) = -2\cos t \Rightarrow v(0) = -2\cos(0) = -2$ ms^{-1}

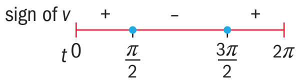
b $-2\cos t = 0$ for $0 < t < \pi \Rightarrow t = \frac{\pi}{2}$ s

c $s\left(\frac{\pi}{2}\right) = 1 - 2\sin\left(\frac{\pi}{2}\right) = 1 - 2 = -1$m

3 a i $v(t)=e^{\sin t}\cos t$ during the interval

$0 \le t \le 2\pi \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

ii



The particle moves left when $v(t)<0$ which is on the interval $\frac{\pi}{2}<t<\frac{3\pi}{2}$

b $v(t)=e^{\sin t}\cos t$

$v'(t)=\left(e^{\sin t}\right)(-\sin t)+(\cos t)\left(e^{\sin t}\right)(\cos t)$

$=-\sin t e^{\sin t}+e^{\sin t}\cos^2 t$

c $s(0)=4$

$s(t)=\int e^{\sin t}\cos t\,dt \Rightarrow u=\sin t;\ \frac{du}{dt}=\cos t$

$s(t)=\int e^{\sin t}\cos t\,dt=\int e^{u}\left(\frac{du}{dt}\right)dt$

$=\int e^{u}du=e^{u}+C=e^{\sin x}+C$

$e^{\sin 0}+C=4 \Rightarrow C=4-1=3$

$s(t)=e^{\sin t}+3$

4 a $v(t)=4\sin t+3\cos t,\ t\ge 0$

displacement after 4 seconds =

$\int_0^4 \left(4\sin t+3\cos t\right)dt$

b $\int_0^4 \left(4\sin t+3\cos t\right)dt \approx 4.34$ m

5 a i $v(t)=-(t+1)\sin\left(\frac{t^2}{2}\right)$ where $t\ge 0$

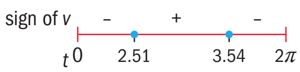
Use a GDC to evaluate $a(1.5)$

$a(1.5)=v'(1.5)\approx -2.52\ \text{ms}^{-2}$

ii $v(1.5)=-2.26$ m

Since velocity and acceleration are both negative at 1.5 seconds, the particle is speeding up.

b $-(t+1)\sin\left(\frac{t^2}{2}\right)=0$ for $0<t<4 \Rightarrow t=2.51, 3.54$



The particle changes direction when velocity changes sign at 2.51 s and 3.54 s.

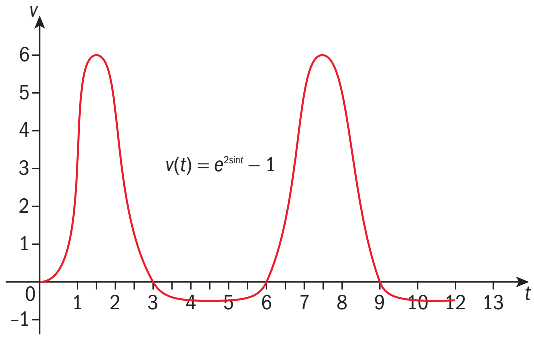
c $\int_0^4 \left|-(t+1)\sin\left(\frac{t^2}{2}\right)\right|dt \approx 7.37$ m

6 a $v(t)=e^{2\sin t}-1$ for $0\le t\le 12$

Use a GDC to evaluate $a(1)$

$a(1)=v'(1)=5.82\ \text{ms}^{-2}$

b i $v(t)=e^{2\sin t}-1$



ii $e^{2\sin t}-1=5 \Rightarrow t\approx 1.11$ s, 2.03 s, 7.39 s, 8.31 s.

iii No, the particle does not return to the origin. Looking at the area between the curve and the t-axis, there is more area above the axis than below indicating that the particle moves to the right a greater distance than to the left, so it never returns to the origin.

c $\int_0^{12}\left|e^{2\sin t}-1\right|dt \approx 24.1$ m

Review exercise

1 a $f(x)=\cos(1-2x)$

$f'(x)=\left[-\sin(1-2x)\right](-2)=2\sin(1-2x)$

b $y=\sin^3 x=(\sin x)^3$

$y'=3(\sin x)^2(\cos x)=3\sin^2 x\cos x$

c $s(t)=e^{\tan t}$

$s'(t)=e^{\tan t}\left(\frac{1}{\cos^2 t}\right)=\frac{e^{\tan t}}{\cos^2 t}$

d $f(x)=\sqrt{\sin x^2}=\left(\sin\left(x^2\right)\right)^{\frac{1}{2}}$

$f'(x)=\left[\frac{1}{2}\left(\sin\left(x^2\right)\right)^{-\frac{1}{2}}\right]\left[\left(\cos(x^2)\right)(2x)\right]=\frac{x\cos x^2}{\sqrt{\sin x^2}}$

e $f(x)=x^2\cos x$

$f'(x)=(x^2)(-\sin x)+(\cos x)(2x)$

$=-x^2\sin x+2x\cos x$

f $y=\ln(\tan x)$

$y'=\left(\frac{1}{\tan x}\right)\left(\frac{1}{\cos^2 x}\right)$ or $\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\cos^2 x}\right)=\frac{1}{\sin x\cos x}$

g $f(x)=(\ln x)(\sin x)$

$f'(x)=(\ln x)(\cos x)+(\sin x)\left(\frac{1}{x}\right)$

$=(\ln x)(\cos x)+\frac{\sin x}{x}$

h $y=2\sin x\cos x$ or $y=\sin 2x$

$y'=(2\sin x)(-\sin x)+(\cos x)(2\cos x)$

$=-2\sin^2 x+2\cos^2 x$

or $y'=(\cos 2x)(2)=2\cos 2x$

2 a $\int(4x^3-\sin x)\mathrm{d}x=4\left(\frac{1}{4}x^4\right)-(-\cos x)+C$

$=x^4+\cos x+C$

b $\int\cos(3x)\mathrm{d}x=\frac{1}{3}\sin(3x)+C$

c $\int\sin(4x+1)\mathrm{d}x=\frac{1}{4}(-\cos(4x+1))+C$

$=-\frac{1}{4}\cos(4x+1)+C$

d $\int x\cos(2x^2)\mathrm{d}x\Rightarrow u=2x^2;\frac{du}{dx}=4x\Rightarrow\frac{1}{4}\left(\frac{du}{dx}\right)=x$

$\int x\cos(2x^2)\mathrm{d}x=\int\frac{1}{4}\left(\frac{du}{dx}\right)\cos(u)\mathrm{d}x$

$=\frac{1}{4}\int\cos u\mathrm{d}u=\frac{1}{4}\sin u+C$

$=\frac{1}{4}\sin(2x^2)+C$

e $\int\frac{\sin(2t+1)}{\cos^2(2t+1)}\mathrm{d}t=\int\frac{1}{\cos^2(2t+1)}(\sin(2t+1))\mathrm{d}t\Rightarrow$

$u=\cos(2t+1);$

$\frac{\mathrm{d}u}{\mathrm{d}t}=-2\sin(2t+1)\Rightarrow-\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)$

$=\sin(2t+1)$

$\int\frac{\sin(2t+1)}{\cos^2(2t+1)}\mathrm{d}t=\int\frac{1}{\cos^2(2t+1)}(\sin(2t+1))\mathrm{d}t$

$=\int\frac{1}{u^2}\left(-\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)\right)\mathrm{d}t$

$=-\frac{1}{2}\int u^{-2}\mathrm{d}u=-\frac{1}{2}\left(\frac{1}{-1}u^{-1}\right)+C$

$=\frac{1}{2\cos(2t+1)}+C$

f $\int\frac{\sin(\ln x)}{x}\mathrm{d}x=\int\sin(\ln x)\left(\frac{1}{x}\right)\mathrm{d}x\Rightarrow$

$u=\ln x;\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\int\frac{\sin(\ln x)}{x}\mathrm{d}x=\int\sin(\ln x)\left(\frac{1}{x}\right)\mathrm{d}x$

$=\int\sin(u)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$

$=\int\sin(u)\mathrm{d}u=-\cos u+C$

$=-\cos(\ln x)+C$

g $\int xe^{\sin x^2}\cos x^2\mathrm{d}x=\int e^{\sin x^2}(x\cos x^2)\mathrm{d}x\Rightarrow$

$u=\sin x^2;$

$\frac{\mathrm{d}u}{\mathrm{d}x}=(\cos x^2)(2x)\Rightarrow$

$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)=x\cos x^2$

$\int xe^{\sin x^2}\cos x^2\mathrm{d}x=\int e^{\sin x^2}(x\cos x^2)\mathrm{d}x$

$=\int e^u\left(\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)\mathrm{d}x$

$=\frac{1}{2}\int e^u\mathrm{d}u=\frac{1}{2}e^u+C$

$=\frac{1}{2}e^{\sin x^2}+C$

h $\int\frac{6\cos x}{(2+\sin x)^2}\mathrm{d}x=6\int\frac{1}{(2+\sin x)^2}(\cos x)\mathrm{d}x\Rightarrow$

$u=2+\sin x;\ \frac{du}{dx}=\cos x$

$\int\frac{6\cos x}{(2+\sin x)^2}\mathrm{d}x=6\int\frac{1}{(2+\sin x)^2}(\cos x)\mathrm{d}x$

$=6\int\frac{1}{u^2}\left(\frac{du}{dx}\right)\mathrm{d}x$

$=6\int u^{-2}du=6(-u^{-1})+C$

$=\frac{-6}{2+\sin x}+C$

3 a $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sin x\,\mathrm{d}x=[-\cos x]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$=-\cos\left(\frac{\pi}{3}\right)+\cos\left(-\frac{\pi}{3}\right)$

$=-\frac{1}{2}+\frac{1}{2}=0$

b $\int_0^{\pi}(1+\sin x)\mathrm{d}x=[x-\cos x]_0^{\pi}$

$=[\pi-\cos(\pi)]-[0-\cos(0)]$

$=(\pi+1)-(0-1)=2+\pi$

c $\int_0^{\pi}(\sin x+\cos 2x)\mathrm{d}x=\left[-\cos x+\frac{1}{2}\sin 2x\right]_0^{\pi}$

$=\left[-\cos(\pi)+\frac{1}{2}\sin(2\pi)\right]-\left[-\cos(0)+\frac{1}{2}\sin(0)\right]$

$=(1+0)-(-1+0)=2$

d $\int_0^{\frac{\pi}{2}}5\sin^{\frac{3}{2}}x\cos x\mathrm{d}x\Rightarrow u=\sin x;\ \frac{du}{dx}=\cos x;$

when $x=0$ then $u=0$ and when $x=\frac{\pi}{2}$

then $u=1$

$$\int_0^{\frac{\pi}{2}} 5\sin^{\frac{3}{2}} x\cos x\,dx = 5\int_{x=0}^{x=\frac{\pi}{2}} u^{\frac{3}{2}}\left(\frac{du}{dx}\right)dx$$
$$= 5\int_{u=0}^{u=1} u^{\frac{3}{2}}du$$
$$= 5\left[\frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$
$$= 5\left[\frac{2}{5}(1)^{\frac{5}{2}} - \frac{2}{5}(0)^{\frac{5}{2}}\right]$$
$$= 2$$

4 $y = \cos(3x-6)$

$y(2) = \cos(0) = 1$

$y' = -\frac{1}{3}\sin(3x-6)$

$m_{\tan} = y'(2) = -\frac{1}{3}\sin(0) = 0$

Therefore at (2, 1) the tangent line is horizontal, so the normal line is the vertical line through (2, 1) or the line $x = 2$

5 Tangent line parallel to $y = \frac{1}{4}x + 3 \Rightarrow m = \frac{1}{4}$

$y = \sin\left(\frac{x}{2}\right), 0 \le x \le \pi \Rightarrow y' = \frac{1}{2}\cos\left(\frac{x}{2}\right), 0 \le x \le \pi$

$\frac{1}{2}\cos\left(\frac{x}{2}\right) = \frac{1}{4} \Rightarrow \cos\left(\frac{x}{2}\right) = \frac{1}{2} \Rightarrow \frac{x}{2} = \frac{\pi}{3} \Rightarrow x = \frac{2\pi}{3}$

$\sin\left(\frac{1}{2}\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

The point is $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$

6 $f(0) = 2$

$f'(x) = x - \sin x \Rightarrow$

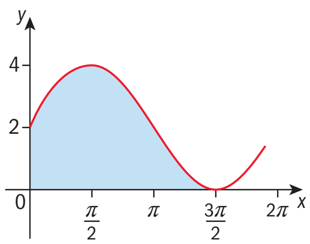
$f(x) = \int(x - \sin x)dx$

$= \frac{1}{2}x^2 + \cos x + C$

$\frac{1}{2}(0)^2 + \cos(0) + C = 2 \Rightarrow C = 1$

$f(x) = \frac{1}{2}x^2 + \cos x + 1$

7 $f(x) = p\sin(x) + q,\ p, q \in \mathbb{N}$



a The sine graph has been stretched by a scale factor of 2 and shifted up 2.

So $p = 2$ and $q = 2$.

b Area $= \int_0^{\frac{3\pi}{2}}(2\sin(x) + 2)dx$

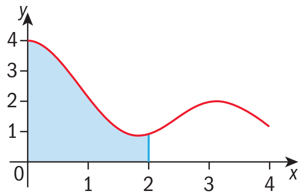
$= [-2\cos x + 2x]_0^{\frac{3\pi}{2}}$

$= \left[-2\cos\left(\frac{3\pi}{2}\right) + 2\left(\frac{3\pi}{2}\right)\right] - [-2\cos(0) + 2(0)]$

$= 3\pi + 2$

Review exercise

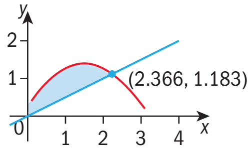
1 a $y = 2\cos^2 x + \cos x + 1$, $x = 0$, $x = 2$ and the x-axis



Area $= \int_0^2 (2\cos^2 x + \cos x + 1)dx \approx 4.53$

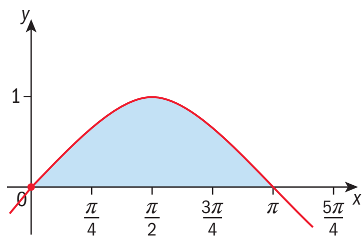
b $y = \sqrt{2\sin x}$ and $y = 0.5x$

$\sqrt{2\sin x} = 0.5x \Rightarrow x = 0, 2.366$



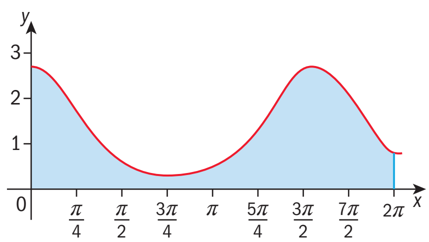
Area $= \int_0^{2.366}\left(\sqrt{2\sin x} - 0.5x\right)dx \approx 1.36$

2 a $y = \sin x$ and the x-axis for $0 \le x \le \pi$



Volume $= \pi\int_0^{\pi}(\sin x)^2\,dx \approx 4.93$

b $y = e^{\cos x}$, $x = 0$ and $x = 2\pi$



Volume $= \pi\int_0^{2\pi}\left(e^{\cos x}\right)^2 dx \approx 45.0$

3 The area under the curve $y = \cos x$ between $x = 0$ and $x = k$, where $0 < k < \frac{\pi}{2}$, is 0.942.

$\int_0^k \cos x dx = 0.942 \Rightarrow [\sin x]_0^k = 0.942 \Rightarrow$

$\sin(k) - \sin(0) = 0.942 \Rightarrow \sin k = 0.942$

$k \approx 1.23$

4 **a** **i** $s(t) = 2e^{\cos(5t)} - 4$

$s'(t) = 2e^{\cos(5t)}(-\sin 5t)(5)$

$= -10\sin(5t)e^{\cos(5t)}$

ii $s''(t) = [-10\sin(5t)]\left[e^{\cos(5t)}(-\sin(5t))(5)\right]$

$+\left[e^{\cos(5t)}\right][-10(\cos(5t))(5)].$

$= 50e^{\cos(5t)}\sin^2(5t) - 50e^{\cos(5t)}\cos(5t)$

$s''(t) = 50e^{\cos(5t)}\left(\sin^2(5t) - \cos(5t)\right)$

iii $s'\left(\frac{\pi}{5}\right) = 0$ and $s''\left(\frac{\pi}{5}\right) \approx 18.4 > 0$

Therefore by the second derivative test s has a relative minimum at $t = \frac{\pi}{5}$.

b Total distance = $\int_0^2 \left|-10\sin(5t)e^{\cos(5t)}\right| dt$

$= 14.2$ m

15 Probability distributions

Answers

Skills check

1 a $\overline{x} = \frac{\sum fx}{\sum f} = \frac{(3\times3)+(4\times5)+(5\times7)+(6\times9)+(7\times6)+(8\times2)}{3+5+7+9+6+2}$

$= \frac{176}{32} = 5.5$

b $\overline{x} = \frac{(10\times3)+(12\times10)+(15\times15)+(17\times9)+(20\times2)}{3+10+15+9+2}$

$= \frac{568}{39} = 14.6$ (3 sf)

2 a $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6\times5}{2} = 15$

b $\binom{8}{5} = \frac{8!}{5!3!} = \frac{8\times7\times6}{3\times2\times1} = 56$

c $\binom{9}{6}(0.3)^3(0.7)^6 = 0.267$

3 a $\frac{5.5}{x} = 3.2 \quad \therefore 5.5 = 3.2x \quad \therefore x = \frac{5.5}{3.2} = 1.71875$

b $\frac{x-2.5}{1.2} = 0.4 \quad \therefore x - 2.5 = 0.48 \quad \therefore x = 2.98$

c $\frac{9-x}{0.2} = 1.6 \quad \therefore 9 - x = 0.32 \therefore x = 9 - 0.32 = 8.68$

Exercise 15A

1 a discrete

b continuous

c discrete

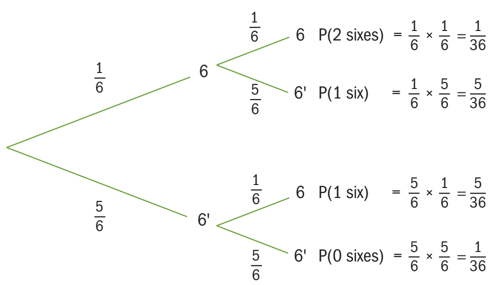
b continuous

2 a

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

s	2	3	4	5	6	7	8	9	10	11	12
P(S = s)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b



$P(2 \text{ sixes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$P(1 \text{ six}) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

$P(1 \text{ six}) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

$P(0 \text{ sixes}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

n	0	1	2
P(*N* = *n*)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

c

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

n	1	2	3	4	5	6
P(*N* = *n*)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

d

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	8	9	12	15	18
4	4	10	12	16	20	24
5	5	12	15	20	25	30
6	6	14	18	24	30	36

p	1	2	3	4	5	6	8	9	10
P(*P* = *p*)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
p	12	15	16	18	20	24	25	30	36
P(*P* = *p*)	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

3 a The faces are numbered 1, 2, 2, 3, 3, 3

	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

t	2	3	4	5	6
$P(T = t)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{12}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

b $P(T > 4) = \frac{12}{36} + \frac{9}{36} = \frac{21}{36} = \frac{7}{12}$

4 a

no. on dice	1	2	3	4	5	6
s	2	1	6	2	10	3

s	1	2	3	6	10
$P(S = s)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b $P(S > 2) = \frac{3}{6} = \frac{1}{2}$

5 a $\frac{1}{3} + \frac{1}{3} + c + c = 1$

$2c = \frac{1}{3} \quad \therefore\ c = \frac{1}{6}$

b $P(1 < X < 4) = P(X = 2) + P(X = 3) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

6 $P(Y = y) = cy^3 \quad y = 1, 2, 3$

y	1	2	3
$P(Y = y)$	c	$8c$	$27c$

$36c = 1 \quad \therefore\ c = \frac{1}{36}$

7 $2k + 4k^2 + 6k^2 + k = 1$

$10k^2 + 3k - 1 = 0$

$(5k - 1)(2k + 1) = 0$

$k = \frac{1}{5}$ (k cannot be negative)

8 $P(X = x) = k\left(\frac{1}{3}\right)^{x-1} \quad x = 1, 2, 3, 4$

x	1	2	3	4
$P(X = x)$	k	$\frac{1}{3}k$	$\frac{1}{9}k$	$\frac{1}{27}k$

$k + \frac{1}{3}k + \frac{1}{9}k + \frac{1}{27}k = 1$

$\frac{40}{27}k = 1 \quad \therefore\ k = \frac{27}{40}$

9 a

x	0	1	2	3	4	5
$P(X = x)$	a	a	a	b	b	b

$3a + 3b = 1 \quad (1) \qquad P(X \geq 2) = 3P(X < 2)$

$a + 3b = 6a$

$3b = 5a \quad (2)$

substitute (2) and (1) $\quad 3a + 5a = 1$

$a = \frac{1}{8} \qquad b = \frac{5}{24}$

b $P(5, 3) = \frac{5}{24} \times \frac{5}{24} = \frac{25}{576} \quad P(3, 5) = \frac{25}{576}$

$P(5, 4) = \frac{25}{576} \quad P(4, 5) = \frac{25}{576} \quad P(5, 5) = \frac{25}{576}$

$P(\text{sum} > 7) = \frac{125}{576}$

10 a $P(C = 3) = P(A = 1 \text{ and } B = 2) + P(A = 2 \text{ and } B = 1)$

$= \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{6} = \frac{2}{9} + \frac{1}{18} = \frac{5}{18}$

b $P(C = 2) = P(A = 1 \text{ and } B = 1) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$P(C = 4) = P(A = 1 \text{ and } B = 3) + P(A = 2 \text{ and } B = 2) + P(A = 3 \text{ and } B = 1)$

$= \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{6} = \frac{6}{18}$

$P(C = 5) = P(A = 2 \text{ and } B = 3) + P(A = 3 \text{ and } B = 2)$

$= \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{2}{3} = \frac{5}{18}$

$P(C = 6) = P(A = 3 \text{ and } B = 3) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

c	2	3	4	5	6
$P(C = c)$	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{6}{18}$	$\frac{5}{18}$	$\frac{1}{18}$

Investigation – dice scores

1

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

d	0	1	2	3	4	5
$P(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

2

d	0	1	2	3	4	5
Expected frequency	6	10	8	6	4	2

3 Mean $= \frac{(0 \times 6) + (1 \times 10) + (2 \times 8) + (3 \times 6) + (4 \times 4) + (5 \times 2)}{36}$

$= \frac{70}{36} = \frac{35}{18}$

4

d	0	1	2	3	4	5
Expected frequency	$\frac{150}{9}$	$\frac{250}{9}$	$\frac{200}{9}$	$\frac{150}{9}$	$\frac{100}{9}$	$\frac{50}{9}$

Mean =

$$\frac{\left(0 \times \frac{150}{9}\right) + \left(1 \times \frac{250}{9}\right) + \left(2 \times \frac{200}{9}\right) + \left(3 \times \frac{150}{9}\right) + \left(4 \times \frac{100}{9}\right) + \left(5 \times \frac{50}{9}\right)}{100}$$

$= \frac{35}{18}$

5 The means are the same

6 $\frac{35}{18}$

Exercise 15B

1

x	1	4	9	16	24	36
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = \left(1\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(9\times\frac{1}{6}\right)+\left(16\times\frac{1}{6}\right)+\left(25\times\frac{1}{6}\right)+\left(36\times\frac{1}{6}\right)$

$= \frac{91}{6} = 15.2$ (3 sf)

2 $\frac{3}{6} + x + y = 1 \quad \therefore x + y = \frac{1}{2} \quad (1)$

$E(Z) = 5\frac{2}{3} \therefore$

$\left(2\times\frac{1}{6}\right) + \left(3\times\frac{1}{6}\right) + \left(5\times\frac{1}{6}\right) + 7x + 11y = \frac{17}{3}$

$7x + 11y = 4 \quad (2)$

Solving (1) and (2), $x = \frac{3}{8}, \quad y = \frac{1}{8}$

3

x	1	2	3	5	8	13
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = \frac{1}{6}(1 + 2 + 3 + 5 + 8 + 13) = \frac{16}{3}$ or $5\frac{1}{3}$

4

x	1	2	3	4	5	6	7	8
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{7}{36}$	$\frac{8}{36}$

$E(X) = \frac{1}{36} + \frac{4}{36} + \frac{9}{36} + \frac{16}{36} + \frac{25}{36} + \frac{36}{36} + \frac{49}{36} + \frac{64}{36}$

$= \frac{204}{36} = 5\frac{2}{3}$

5 a

x	1	2	3	4	5	6	7	8	9
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$	$4k$	$3k$	$2k$	k

$25k = 1 \quad \therefore k = \frac{1}{25}$

b From symmetry, $E(X) = 5$

6 a

x	1	2	3
$P(X = x)$	0.2	$1 - k$	$k - 0.2$

b $0 \le 1 - k \le 1$ and $0 \le k - 0.2 \le 1$

$-1 \le -k \le 0 \qquad 0.2 \le k \le 1.2$

$1 \ge k \ge 0$

$\therefore 0.2 \le k \le 1$

c Mean $= 0.2 + 2(1 - k) + 3(k - 0.2)$

$= 0.2 + 2 - 2k + 3k - 0.6 = k + 1.6$

7

x	1	2	4
$P(x = x)$	a	0.3	b

$a + b = 0.7 \quad (1)$

mean $= 2.8 \qquad \therefore a + 0.6 + 4b = 2.8$

$a + 4b = 2.2 \quad (2)$

solving (1) and (2), $a = 0.2, b = 0.5$

$\therefore P(x = 1) = 0.2$

8 a $P(R = 1) = \frac{2}{10} = \frac{1}{5}$

$P(R = 2) = \frac{8}{10}\times\frac{2}{9} = \frac{16}{90} = \frac{8}{45}$

$P(R = 3) = \frac{8}{10}\times\frac{7}{9}\times\frac{2}{8} = \frac{14}{90} = \frac{7}{45}$

$P(R = 4) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{2}{7} = \frac{12}{90} = \frac{6}{45}$

$P(R = 5) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{2}{6} = \frac{10}{90} = \frac{5}{45}$

$P(R = 6) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{2}{5} = \frac{8}{90} = \frac{4}{45}$

$P(R = 7) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4} = \frac{6}{90} = \frac{3}{45}$

$P(R = 8) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3} = \frac{4}{90} = \frac{2}{45}$

$P(R = 9) = \frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}$

$= \frac{2}{90} = \frac{1}{45}$

r	1	2	3	4	5	6	7	8	9
$P(R = r)$	$\frac{9}{45}$	$\frac{8}{45}$	$\frac{7}{45}$	$\frac{6}{45}$	$\frac{5}{45}$	$\frac{4}{45}$	$\frac{3}{45}$	$\frac{2}{45}$	$\frac{1}{45}$

b Mean $= \frac{1}{45}(9 + 16 + 21 + 24 + 25 + 24 + 21$

$+ 16 + 9) = 3\frac{2}{3}$

9 a $P(R = 2) = \frac{8}{10}\times\frac{2}{10} = \frac{16}{100} = \frac{4}{25}$

b $P(R = 3) = \left(\frac{8}{10}\right)^2\times\frac{2}{10} = \frac{16}{125}$

c $P(R = n) = 0.8^{n-1}\times 0.2$

d 1

10 a $P(Z = 0) + 0.2 + 0.05 + 0.001 + 0.0001 = 1$

$P(Z = 0) = 1 - 0.2511$

$= 0.7489$

b $E(Z) = 0 + 0.4 + 1 + 0.2 + 0.1$

$= 1.7$

\$1.70 is the expected winnings on a ticket

c A ticket costs \$2, but you only expect to win \$1.70. Therefore you expect to lose \$0.30

Investigation – the binomial quiz

You would expect to get 2.5 questions right

$P(3 \text{ right}) = \binom{5}{3}(0.5)^3(0.5)^2 = 0.3125$

Exercise 15C

1 $X \sim B\left(4, \frac{1}{2}\right)$

a $P(X = 1) = \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = \frac{1}{4}$

b $P(X < 1) = P(X = 0) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

c $P(X \le 1) = P(X = 0 \text{ or } 1) = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$

d $P(X \ge 1) = 1 - P(X = 0) = 1 - \frac{1}{16} = \frac{15}{16}$

2 using a GDC:

a 0.329 **b** 0.351

c 0.680 **d** 0.649

3 using a GDC:

a $P(X = 5) = 0.0389$

b $P(X < 5) = 0.952$

c $P(X > 5) = 0.00870$

d $P(X \geq 1) = 0.932$

Exercise 15 D

(Using a GDC where possible)

1 $X \sim B(4, 0.25)$

x	0	1	2	3	4
$P(X = x)$	0.316	0.422	0.211	0.0469	0.00391

Most likely outcome is 1 red face with probability 0.422

2 $X \sim B(8, 0.55)$

a $P(X = 5) = 0.257$

b P(at least 5 times) $= P(X < 3) = 0.260$

3 $X \sim B(16, 0.01)$

a $P(X = 0) = 0.851$

b P(13 not faulty) $= P(X = 3) = 0.000491$

c $P(X \geq 2) = 0.0109$

4 $X \sim B(10, 0.25)$

a $P(X = 5) = 0.0584$

b P(at least 3 free) $= P(X \leq 7) = 0.9996$

5 $X \sim B(5, 0.4)$

$P(X \leq 3) = 0.913$

6 $X \sim B(6, 0.15)$

a $P(X > 1) = 0.224$

b $P(X = 1) = 0.399$

7 $X \sim B(15, 0.05)$

a **i** $P(X = 3) = 0.0307$

ii $P(X = 0) = 0.463$

iii $P(X \geq 2) = 0.171$

b **i** $(0.46329\ldots)^2 = 0.215$

ii $(0.17095\ldots.)^2 = 0.0292$

iii $0.46329\ldots.. \times 0.17095\ldots. \times 2 = 0.158$

Exercise 15 E

1 $X \sim B(n, 0.6)$ $\quad P(X < 1) = 0.0256$

$P(X = 0) = 0.0256$

$(0.4)^n = 0.0256$

$n \log 0.4 = \log 0.0256$

$n = 4$

2 $X \sim B(n, 0.01)$ $\quad P(x = 0) > 0.5$

$0.99^n > 0.5$

$n \log 0.99 > \log 0.5$

$n < \frac{\log 0.5}{\log 0.99} = 68.9$, so the largest sample size is 68

3 $X \sim B(n, 0.2)$ $\quad P(X \geq 1) > 0.75$

$1 - P(X = 0) > 0.75$

$1 - 0.8^n > 0.75$

$0.25 > 0.8^n$

$0.8^n < 0.25$

$n \log 0.8 < \log 0.25$

$n > \frac{\log 0.25}{\log 0.8}$

$n > 6.21$

$\therefore$ least value of $n = 7$

4 $X \sim B(n, 0.3)$ $\quad P(x \geq 1) > 0.95$

$1 - P(x = 0) > 0.95$

$1 - 0.7^n > 0.95$

$0.05 > 0.7^n$

$0.7^n < 0.05$

$n \log 0.7 < \log 0.05$

$n > \frac{\log 0.05}{\log 0.7}$

$n > 8.399$

$\therefore$ least number of attempts is 9

5 $X \sim B(n, 0.5)$ $\quad P(X \geq 1) > 0.99$

$1 - P(X = 0) \geq 0.99$

$1 - 0.5^n \geq 0.99$

$0.01 \geq 0.5^n$

$0.5^n \leq 0.01$

$n \log 0.5 \leq \log 0.01$

$n \geq \frac{\log 0.01}{\log 0.5}$

$n > 6.64$ so the coin must be tossed 7 times.

Exercise 15F

1 **a** $X \sim B(40, 0.5)$ $\quad E(X) = 40 \times 0.5 = 20$

b $X \sim B\left(40, \frac{1}{6}\right)$ $\quad E(X) = 40 \times \frac{1}{6} = 6\frac{2}{3}$

c $X \sim B(40, 0.25)$ $\quad E(X) = 40 \times 0.25 = 10$

2 $X \sim B(n, p)$ $\quad$ mean $= 10$ $\quad p = 0.4$ $\quad np = 10$

$n \times 0.4 = 10$ $\quad \therefore n = 25$

3 **a** $X \sim B(15, 0.25)$

b mean $= 15 \times 0.25 = 3.75$

c $P(X \geq 10) = 0.000795$

4 **a** total number of girls =
$(1 \times 34) + (2 \times 40) + (3 \times 13) = 158$
total number of children $= 100 \times 3 = 300$
$\therefore$ P(girl) $= \frac{153}{300} = 0.51$

b $X \sim B(3, 0.51) \quad P(x = 2) = 0.382$
expected number of families $= 0.382 \times 100$
$= 38.2$

Exercise 15G

1 $X \sim B\left(0, \frac{1}{4}\right)$
Mean $= 0 \times \frac{1}{4} = 0$
Variance $= 0 \times \frac{1}{4} \times \left(1 - \frac{1}{4}\right) = 0$

2 B(12, 0.6)
Mean $= 12 \times 0.6 = 7.2$
Variance $= 12 \times 0.6 \times 0.4 = 2.88$
Standard deviation $= \sqrt{2.88} = 1.70$ (3 sf)

3 $X \sim B\left(40, \frac{1}{2}\right)$
Mean $= 40 \times \frac{1}{2} = 20$
Variance $= 40 \times \frac{1}{2} \times \frac{1}{2} = 10$
Standard deviation $= \sqrt{10} = 3.16$ (3 sf)

4 $X \sim B\left(10, \frac{1}{6}\right)$
a $E(X) = 10 \times \frac{1}{6} = \frac{5}{3}$
b $Var(X) = 10 \times \frac{1}{6} \times \frac{5}{6} = \frac{25}{18}$
c $P(X < \mu) = P\left(X < \frac{5}{3}\right) = 0.485$ (3 sf)
(using binomial CDF on the GDC)

5 $X \sim B\left(22, \frac{1}{5}\right)$
a $E(X) = 22 \times \frac{1}{5} = \frac{22}{5}$
b $Var(X) = 22 \times \frac{1}{5} \times \frac{4}{5} = \frac{88}{25}$
c $P(X < 4) = 0.332$ (3 sf) using binomial CDF on the GDC)

6 $X \sim B(n, p)$
$E(X) = 4.5$, $Var(X) = 3.15$
$E(X) = np = 4.5$
$Var(X) = npq = np(1 - p) = 3.15$
$4.5(1 - p) = 3.15$
$1 - p = \frac{3.15}{4.5}$
$= 0.7$
$p = 1 - 0.7$
$= 0.3$
$np = 4.5$
$n = \frac{4.5}{p} = \frac{4.5}{0.3} = 15$
$P(X \geq 3) = 1 - P(X < 3) = 1 - 0.126828$
$= 0.873$ (3 sf)

7 $X \sim B(n, p)$
$E(X) = 7.8$
$p = 0.3$
a $E(X) = np = 7.8$
$0.3n = 7.8$
$n = \frac{7.8}{0.3}$
$n = 26$
b $Var(X) = npq = np(1 - p)$
$= 26 \times 0.3 \times 0.7$
$= 5.46$

8 $X \sim B(n, p)$
$E(X) = 9.6$
$Var(X) = 1.92$
$E(X) = np = 9.6$
$Var(X) = npq = np(1 - p) = 1.92$
$9.6(1 - p) = 1.92$
$9.6 - 9.6p = 1.92$
$9.6p = 7.68$
$p = 0.8$
$n = \frac{9.6}{0.8}$
$n = 12$
$P(X = 6) = 0.0155$ (using binomial PDF on the GDC)

Exercise 15H

Using GDC:

1 **a** 0.683
b 0.954
c 0.997

2 **a** $P(1 < Z < 2) + P(-2 < Z < -1) = 0.272$
b $P(0.5 < Z < 1.5) + P(-1.5 < Z < -0.5)$
$= 0.483$

3 **a** $P(Z > 1) = 0.159$
b $P(Z > 2.4) = 0.00820$

4 **a** $P(Z < -1) = 0.159$
b $P(Z < -1.75) = 0.0401$

5 **a** 0.742
b 0.236
c 0.0359
d 0.977
e 0.390

6 **a** 0.306
b 0.595
c 0.285

7 **a** $P(|Z| < 0.4) = P(-0.4 < Z < 0.4) = 0.311$

b $P(|Z| > 1.24) = 1 - P(|Z| < 1.24)$
$= 1 - P(-1.24 < Z < 1.24)$
$= 0.215$

Exercise 15I

Using GDC:

1 **a** 0.655
b 0.841
c 0.186
d 0.5

2 **a** 0.672
b 0.748
c 0.345

3 **a** 0.994
b 0.997
c 0.494

Exercise 15J

1 $X \sim N(100, 20^2)$

a $P(X < 130) = 0.933$

b $P(X > 90) = 0.691$

c $P(80 < X < 125) = 0.736$

2 $X \sim N(4, 0.25^2)$

$P(3.5 < X < 4.5) = 0.9545$

number of acceptable bolts $= 0.9545 \times 500 = 477$

3 $X \sim N(14, 4^2)$

a $P(X > 20) = 0.0668$

b $P(X > 10) = 0.159 = 15.9\%$

4 $X \sim N(551.3, 15^2)$

$P(X > 550) = 0.535 = 53.6\%$

5 $X \sim N(500, 20^2)$

a $P(X < 475) = 0.106$

b $(0.1056\ \ldots)^3 = 0.00118$

Exercise 15K

1 **a** $P(Z < a) = 0.922$, $a = 1.42$

b $P(Z > a) = 0.342$
$\therefore P(Z < a) = 0.658$, $a = 0.407$

c $P(Z > a) = 0.005 \quad \therefore P(Z < a) = 0.995$,
$a = 2.58$

2 **a** $P(1 < Z < a) = 0.12$
$P(Z < 1) = 0.8413 \quad \therefore P(Z < a) = 0.9613$
$\therefore a = 1.77$

b $P(a < Z < 1.6) = 0.787$
$P(Z > 1.6) = 0.0548$
$\therefore P(Z < a) = 1 - (0.787 + 0.0548)$
$= 0.1582$
$\therefore a = -1.00$

c $P(a < Z < -0.3) = 0.182$
$P(Z > -0.3) = 0.6179$
$\therefore P(Z < a) = 1 - (0.182 + 0.6179) = 0.2001$
$\therefore a = -0.841$

3 **a** $P(-a < Z < a) = 0.3$
$\therefore P(Z < a) = 0.65 \quad \therefore a = 0.385$

b $P(|Z| > a) = 0.1096$
$\therefore P(Z < a) = 0.9452 \quad \therefore a = 1.60$

4 **a** $P(Z < a) = 0.95 \quad \therefore a = 1.64$

b $P(Z < a) = 0.8 \quad \therefore a = 0.842$

Exercise 15L

1 $X \sim N(5.5, 0.2^2) \quad P(X > a) = 0.235$
$P(x < a) = 0.765 \quad \therefore a = 5.64$

2 $M \sim N(420, 10^2)$

a $P(M < a) = 0.25 \quad \therefore a = 413$

b $P(M < b) = 0.9 \quad \therefore b = 433$

3 $X \sim N(502, 1.6^2)$

a $P(x < 500) = 0.106$

b $P(500 < x < 505) = 0.864$ or 86.4%

c $P(x < b) = 0.975 \quad b = 505.1 \quad a = 498.9$
$a = 499 \quad b = 505$

4 $X \sim N(550, 25^2)$

a $P(520 < X < 570) = 0.673$

b $P(X > a) = 0.1 \quad \therefore P(X < a) = 0.9 \quad \therefore a = 582$

5 **a** $X \sim N(55, 15^2)$, $P(x > d) = 0.05$, $d = 79.7$

b $P(x < f) = 0.90$, $f = 35.8$

Exercise 15M

1 $X \sim N(30, \sigma^2) \quad P(X > 40) = 0.115$

$Z = \frac{40-30}{\sigma} = \frac{10}{\sigma}$

$\therefore \frac{10}{\sigma} = 1.2004 \quad \therefore \sigma = 8.33$

2 $X \sim N(\mu, 4^2) \quad P(X < 20.5) = 0.9$

$Z = \frac{20.5-\mu}{4} \quad \therefore P\left(Z < \frac{20.5-\mu}{4}\right) = 0.9$

$\therefore \frac{20.5-\mu}{4} = 1.28155$

$\therefore \mu = 15.4$

3 $X \sim N(\mu, \sigma^2) \quad P(X > 58.39) = 0.0217$
$P(X < 41.82) = 0.0287$

$Z = \frac{59.39-\mu}{\sigma} \quad Z = \frac{41.82-\mu}{\sigma}$

$P\left(Z > \frac{58.39-\mu}{\sigma}\right) = 0.0217$

$\therefore P\left(Z < \frac{58.39-\mu}{\sigma}\right) = 0.9783 \quad \therefore \frac{58.39-\mu}{\sigma} = 2.0198$

$P\left(Z < \frac{41.82-\mu}{\sigma}\right) = 0.0287 \quad \therefore \frac{41.82-\mu}{\sigma} = -1.9003$

$\therefore \mu = 49.9 \quad \sigma = 4.23$

4 $X \sim N(\mu, \sigma^2)$ $\quad P(X < 89) = 0.90 \quad P(X < 94) = 0.95$

$Z = \frac{89-\mu}{\sigma} \qquad Z = \frac{94-\mu}{\sigma}$

$P\left(Z < \frac{89-\mu}{\sigma}\right) = 0.90 \quad \therefore \quad \frac{89-\mu}{\sigma} = 1.28155$

$P\left(Z < \frac{94-\mu}{\sigma}\right) = 0.95 \quad \therefore \quad \frac{94-\mu}{\sigma} = 1.64485$

$\therefore \mu = 71.4 \qquad \sigma = 13.8$

5 $X \sim N(136, \sigma^2) \quad P(X > 145) = 0.12$

$Z = \frac{145-130}{\sigma} = \frac{9}{\sigma}$

$P\left(Z > \frac{9}{\sigma}\right) = 0.12 \quad \therefore P\left(Z < \frac{9}{\sigma}\right) = 0.88$

$\therefore \frac{9}{\sigma} = 1.175 \qquad \therefore \sigma = 7.66$ cm

6 $X \sim N(\mu, 20^2) \quad P(X < 500) = 0.01$

$Z = \frac{500-\mu}{20}$

$P\left(Z < \frac{200-\mu}{20}\right) = 0.01 \quad \therefore \quad ^{x+4=\pm\sqrt{19}} = -2.326$

$\therefore \mu = 546.5$ or 547 g

7 $X \sim N(0.85, \sigma^2) \quad P(X < 1.1) = 0.74$

a $Z = \frac{1.1-0.85}{\sigma} = \frac{0.25}{\sigma}$

$P\left(Z < \frac{0.25}{\sigma}\right) = 0.74 \quad \therefore \quad \frac{0.25}{\sigma} = 0.6433$

$\therefore \sigma = 0.389$ kg

b $P(x > 1) = 0.350 = 35.0\%$

8 $X \sim N(\mu, 7^2) \quad P(X > 68) = 0.025$

$Z = \frac{68-\mu}{7}$

$P\left(Z > \frac{68-\mu}{7}\right) = 0.025 \quad \therefore P\left(Z < \frac{68-\mu}{7}\right) = 0.975$

$\therefore \quad \frac{68-\mu}{7} = 1.95996 \qquad \therefore \mu = 54.3$ cm

9 $X \sim N(2.9, \sigma^2) \quad P(X > 3) = 0.35$

$Z = \frac{3-2.9}{5} = \frac{0.1}{\sigma}$

$P\left(Z > \frac{0.1}{\sigma}\right) = 0.35 \quad \therefore P\left(Z < \frac{0.1}{\sigma}\right) = 0.65$

$\therefore \frac{0.1}{\sigma} = 0.3853 \qquad \therefore \sigma = 0.260$ m

10 a $X \sim N(\mu, \sigma^2)\ P(X < 10.8) = 0.3\ P(X > 154) = 0.2$

$Z = \frac{108-\mu}{\sigma} \qquad Z = \frac{154-\mu}{\sigma}$

$P\ x = \frac{-7\pm\sqrt{65}}{2} = 0.3 \quad \therefore \quad \frac{108-\mu}{\sigma} = -0.5244$

$P\left(Z > \frac{154-\mu}{\sigma}\right) = 0.2 \quad \therefore P\left(Z < \frac{154-\mu}{\sigma}\right) = 0.8$

$\therefore \frac{154-\mu}{\sigma} = 0.8416$

$\therefore \mu = 125.7 \qquad \sigma = 33.67$

$\mu = 126 \qquad \sigma = 33.7$

b $P(X > 117) = 0.605 = 60.5\%$

yes, this is consistent with the normal distribution

11 $X \sim N(\mu, \sigma^2) \quad P(X > 495) = 0.95$

$P(X > 490) = 0.99$

$\frac{495-\mu}{\sigma} \qquad \frac{490-\mu}{\sigma}$

$P\left(Z > \frac{495-\mu}{\sigma}\right) = 0.95 \quad \therefore P\left(Z < \frac{495-\mu}{\sigma}\right) = 0.05$

$\therefore \frac{495-\mu}{\sigma} = -1.64485$ (1)

$P\left(Z > \frac{490-\mu}{\sigma}\right) = 0.99 \quad \therefore P\left(Z < \frac{490-\mu}{\sigma}\right) = 0.01$

$\therefore \frac{490-\mu}{\sigma} = -2.32635$ (2)

Solving (1) and (2) simultaneously gives

$\mu = 507.1, \sigma = 7.34$

Review exercise

1 a $0.3 + \frac{1}{k} + \frac{2}{k} + 0.1 + 2.1 = 1$

$\frac{3}{k} = 0.5 \quad \therefore k = 6$

b $E(X) = (-2 \times 0.3) + \left(-1 \times \frac{1}{6}\right) + (1 \times 0.1)$

$+ (2 \times 0.1)$

$= -\frac{7}{15}$

2 a

x	1	2	3	4	5
P(X = x)	5c	8c	9c	8c	5c

$35c = 1$

$\therefore c = \frac{1}{35}$

b from symmetry, $E(X) = 3$

3 $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + x = 1 \quad \therefore x = \frac{3}{8}$

$P(6) = P(2, 4) + P(3, 3) + P(4, 2)$

$= \frac{1}{4} \times \frac{3}{8} + \frac{1}{8} \times \frac{1}{8} + \frac{3}{8} \times \frac{1}{4} = \frac{13}{64}$

4 a

	2	2	4	4
1	2	2	4	4
2	4	4	8	8
3	6	6	12	12
4	8	8	16	16

possible values of P are 2, 4, 6, 8, 12, 16

b

x	2	4	6	8	12	16
P(X = x)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

c $E(P) = \frac{2}{8} + \frac{8}{8} + \frac{6}{8} + \frac{16}{8} + \frac{12}{8} + \frac{6}{8}$

$= 7.5$

d

x	£10	£5
$P(X = x)$	$\frac{1}{4}$	$\frac{3}{4}$

$E(X) = 10 \times \frac{1}{4} + 5 \times \frac{3}{4} = 6.25$

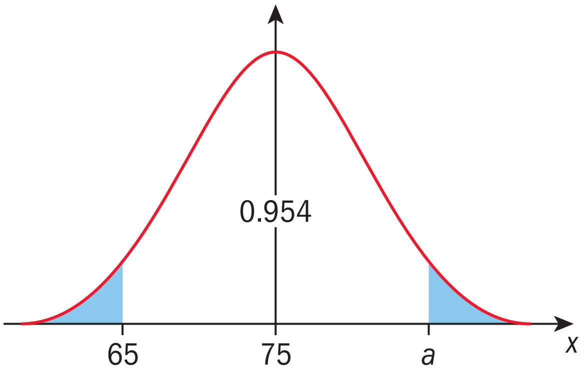
After 10 weeks, expected total $= 6.25 \times 10$
$= £62.50$

5 $X \sim B\left(5, \frac{1}{3}\right)$ $\quad P(X = 3) = \binom{5}{3} \times \left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right)^2$

$= 10 \times \frac{1}{27} \times \frac{4}{9} = \frac{40}{243}$

6 $X \sim B(2, 0.1)$ $\quad E(X) = nP = 2 \times 0.1 = 0.2$

7 a



$P(X < 65) = P(X > a)$

From symmetry, $a = 85$

b $P(X > a) = \frac{1 - 0.954}{2} = 0.023$

Review exercise

1 a $X \sim B\left(3, \frac{1}{3}\right)$ $\quad P(X \geq 1) = \frac{19}{27}$

b

x	-5	1
$P(X = x)$	$\frac{8}{27}$	$\frac{19}{27}$

c i $E(X) = -5 \times \frac{8}{27} + 1 \times \frac{19}{27} = -\$\frac{7}{9}$ or $-\$0.78$

$\therefore$ lose \$0.78 (or $\$\frac{7}{9}$)

ii $\frac{7}{9} \times 9 = 7$ $\quad \therefore$ lose \$7

2 $X \sim B(8, 0.3)$

a $P(X = 3) = 0.254$

b $P(X \geq 3) = 0.448$

3 $X \sim B\left(6, \frac{1}{6}\right)$ $\quad P(X = 3) = 0.05358$

$Y \sim B(5, 0.05358)$ $\quad P(Y = 2) = 0.0243$

4 a $X \sim B(10, 0.2)$

i $P(X = 4) = 0.0881$

ii $P(X > 5) = 0.00637$

b $P(X = 0) = 0.107$ $\quad P(X = 1) = 0.268$

$P(X = 2) = 0.302$

$P(X = 3) = 0.201$ the probabilities continue to decrease after this

$\therefore$ most likely number is 2

c $X \sim B(n, 0.2)$ $\quad P(X \geq 1) > 0.95$

$1 - P(X = 0) > 0.95$

$0.05 > P(X = 0)$

$P(X = 0) < 0.05$

$(0.8)^n < 0.05$

$n \log 0.8 < \log 0.05$

$n > \frac{\log 0.05}{\log 0.8}$

$n > 13.4$

$\therefore$ need 14 points in this sample

5 $P(|Z| \leq a) = 0.85$

$P(-a \leq Z \leq a) = 0.85$

$\therefore P(Z \leq a) = 0.925$ $\quad \therefore a = 1.44$

6 a $X \sim N(71, \sigma^2)$ $\quad p(x < 80) = 0.85$

$Z = \frac{80 - 71}{\sigma} = \frac{9}{\sigma}$

$P\left(Z < \frac{9}{\sigma}\right) = 0.85$ $\quad \therefore \frac{9}{\sigma} = 1.0364$ $\quad \therefore \sigma = 8.68$

b $P(X > 65) = 0.755$

7 $X \sim N(\mu, \sigma^2)$ $\quad P(X < 30) = 0.15$ $\quad P(X > 50) = 0.1$

$Z = \frac{30 - \mu}{\sigma}$ $\quad Z = \frac{50 - \mu}{\sigma}$

$P\left(Z < \frac{30 - \mu}{\sigma}\right) = 0.15$ $\quad \therefore \frac{30 - \mu}{\sigma} = -1.03643$

$P\left(Z > \frac{50 - \mu}{\sigma}\right) = 0.1$ $\quad \therefore P\left(Z > \frac{50 - \mu}{\sigma}\right) = 0.9$

$\therefore \frac{50 - \mu}{\sigma} = 1.28155$

$\mu = 38.9$ hours $\quad \sigma = 8.63$ hours

8 a $X \sim N(\mu, 2)$ $\quad P(x > 35) = 0.2$

$Z = \frac{35 - \mu}{2}$

$P\left(Z > \frac{35 - \mu}{2}\right) = 0.2$ $\quad \therefore P\left(Z < \frac{35 - \mu}{2}\right) = 0.8$

$\therefore \frac{35 - \mu}{2} = 0.8416$

$\therefore \mu = 33.3$

b $X \sim P(5, 0.2)$ $\quad P(X = 0) = 0.328$

c $P(X \geq 2) = 0.263$